Honors ICM- Graph Theory — **Name________________________**

Unit 6 Homework Packet — **Period: _____**

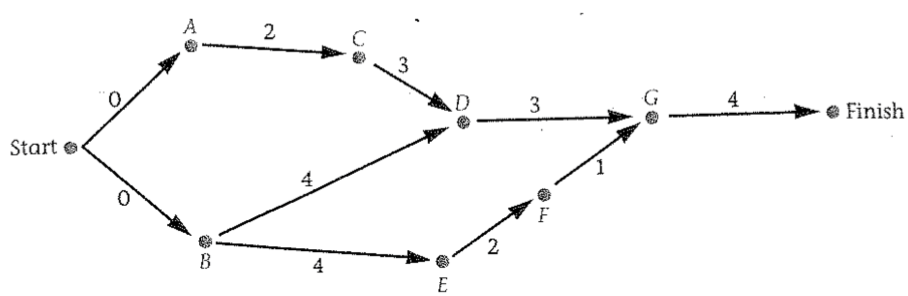
Homework Day 1

5. To help organize the task of completing the family dinner, Mrs. Shu listed the following tasks.

Task	Time (min.)	Prerequisite Task
Start	0	—
A Wash hands		
B Defrost hamburger		
C Shape meat into patties		
D Cook hamburgers		
E Peel and slice potatoes		
F Fry potatoes		
G Make salad		
H Set table		
I Serve food		

a. Complete the table by making reasonable time estimates in minutes for each of these tasks and indicating the prerequisites.
b. Construct a graph using the information from your table.
c. What is the least amount of time needed to prepare dinner?

7. Consider the following graph.



a. Complete the following task table for this graph.

Task	Time	Prerequisite Task
Start	0	—
A		
B		
C		
D		
E		
F		
G		
Finish		

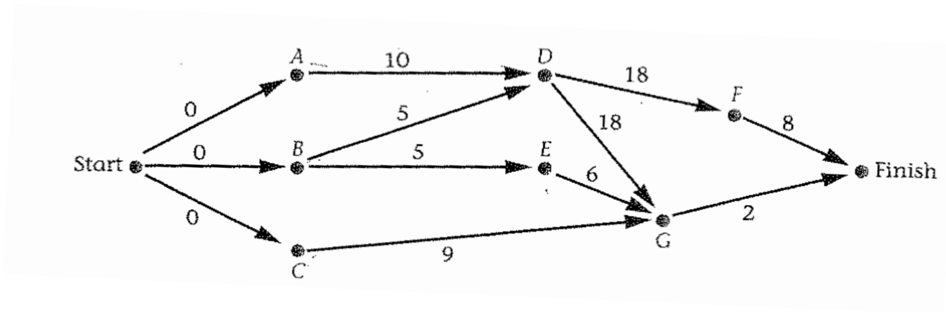
Critical Path:

Project Time:

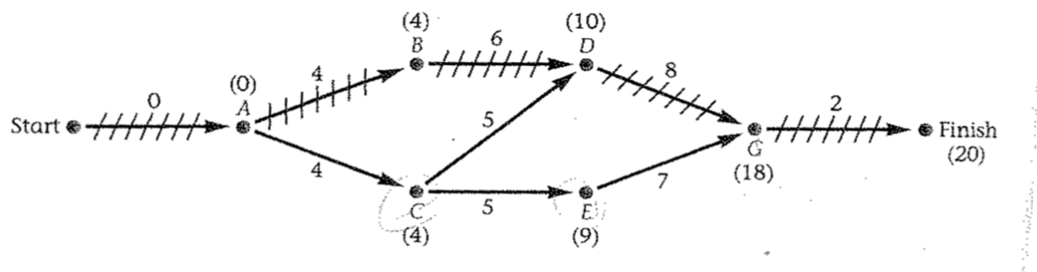
b. What is the least amount of time it will take to complete all of the tasks in this graph? Explain why it cannot be completed in less time.

6. Construct a graph with three critical paths.

7. Determine the minimum project time and the critical path for the following graph.



8. In the following graph, each vertex has been labeled with its EST and the critical path is marked.

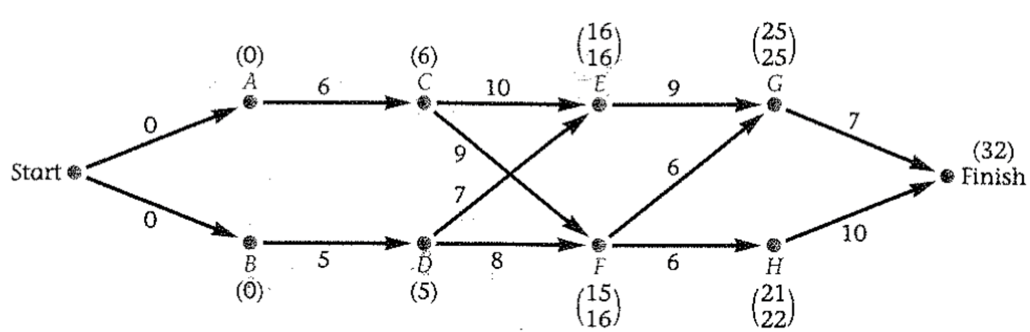


a. Task E can begin as early as day 9. If it begins on day 9, when will it be completed?
If it begins on day 10? On Day 11? What will happen if it beings on day 12?

b. What is the latest day on which task E can begin if task G is to begin on Day 18.

c. What is the Latest Start Time for task C?

9. To find the LST for each task, it is necessary to begin with the Finish and work through the graph in reverse order to the Start. Each of the vertices in the following graph are labeled with their ESTs. The LSTs for several of the tasks have been calculated and are shown below the ESTs on the vertices. Find the LSTs for the remaining tasks.



Homework Day 2-3

3. Draw a graph with vertices = {A, B, C, D, E, F} and edges = {AB, CD, DE, EC, EF}

   a. Name two vertices that are not adjacent.

   b. F, E, C is one possible path from F to C. This path has a length of 2, since two edges were traveled to get from F to C. Name a path from F to C with a length of 3.

   c. Is this graph connected? Explain why or why not.

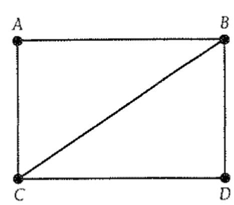
   d. Is this graph complete? Explain why or why not.

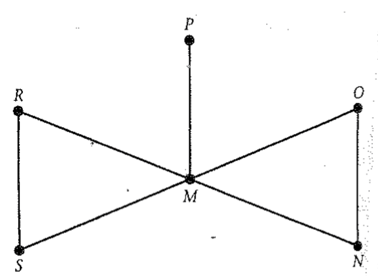
4. Draw a graph with five vertices in which vertex W is adjacent to Y, X is adjacent to Z, and V is adjacent to each of the other vertices.

5. Construct a graph for each adjacency matrix. Label the vertices A, B, C…

a. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

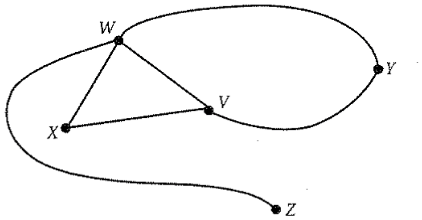
6. Create an adjacency matrix for each of the following graphs:

a.


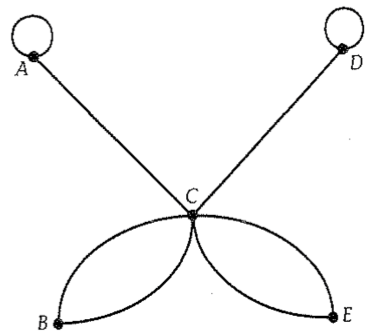
b.


7. If an adjacency matrix has a 1 on the main diagonal, what would that indicate? What would a 2 in row 2, column 1 indicate?

8. In a graph, the number of edges that have a specific vertex as an endpoint is known as the **degree** of the vertex. In the following graph, the degree of vertex W is 4. This is denoted as deg(W)=4. Find the degree of each of the other vertices.



9. An edge that connects a vertex to itself is called a **loop**. When finding the degree of a vertex on which there is a loop, the loop is counted twice. For example deg(A) = 3

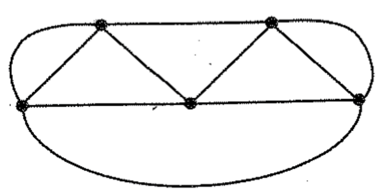


a. Find the degree of vertices B, C, D, and E.

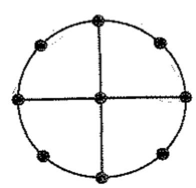
b. Give the adjacency matrix for the above graph.

10. State whether each graph has an Euler circuit, an Euler path or neither. **Explain why.**

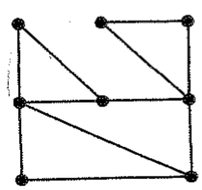
a.



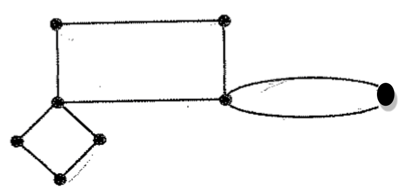
b.



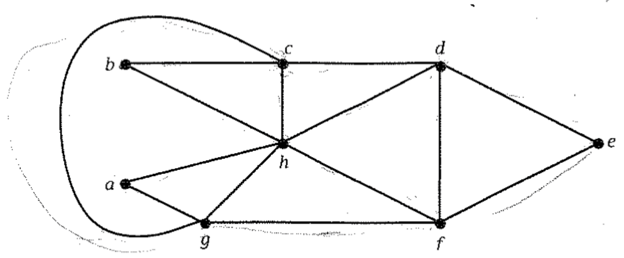
c.



d.



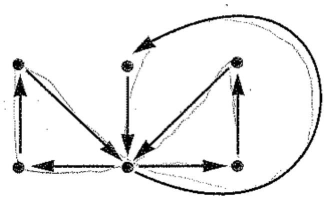
11. Find an Euler circuit for the following graph.



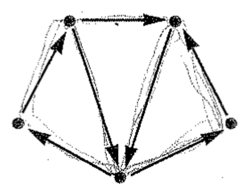
12. Will a complete graph with 2 vertices have an Euler circuit? With 3 vertices? With 4 vertices? With 5 vertices? With n vertices? Explain.

9. Determine whether the digraph has a directed Euler circuit.

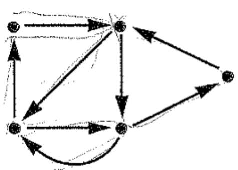
a.



b.



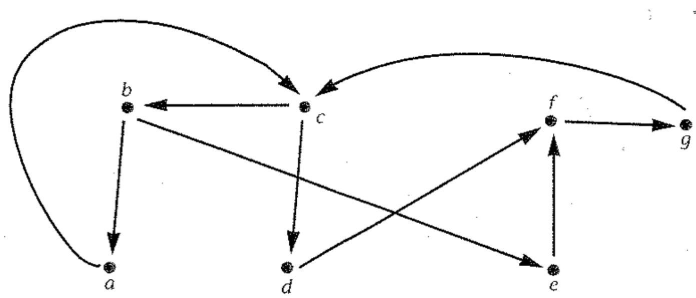
c.



10. a. Does the following digraph have a directed Euler circuit? Explain why or why not.

b. Does it have a directed Euler path? If it does, which vertices can be the starting vertex?

c. Write a general statement explaining when a digraph has a directed Euler path.



4.1-4.5 Practice

1. Use the table below #1-2, showing the steps to produce a padlock, in order to construct a graph. Label each vertex with the earliest start time. Determine the minimum project time and critical path.

Activity	Description	Immediate Predecessor	Duration (Hours)
A	Receive raw materials	-	0.5
B	Bolt cutting	A	1.0
C	Transfer Machine (series of drilling and cutting operations)	B	1.5
D	Transfer Machine (barrels)	B	1.4
E	Barrel pinning	D	1.2
F	Shackle groove cutting	B	0.8
G	Shackle Bending	F	1.0
H	Insert shackle into body	C,E,G	0.4
I	Insert barrel into body and test key set	H	1.4
J	Packaging of padlock	I	0.5

Source: http://criticalpathmethod.weebly.com/solved-problem.html

2. Now find the LST (Latest Start Time) for G and C and list the critical path.

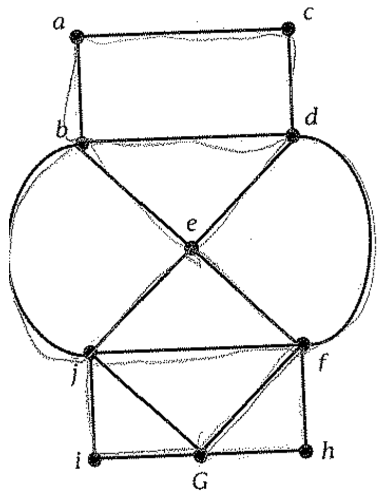
LST G:_______ LST C:________ Critical Path:____________________

3. Central High School is a member of a five-team hockey league. Each team in the league plays exactly two games, which must be against different teams. Show that there is only one possible graph for this schedule.

4. Draw a tournament with three vertices in which:
   a) One player wins all the games he or she plays.
   b) Each player wins exactly one game.
   c) Two players lose all of the games they play.

5. Draw a tournament with five vertices in which there is a three-way tie for first place.

6. The street network of a city can be modeled with a graph in which the vertices represent the street corners, and the edges represent the streets. Suppose you are the city street inspector and it is desirable to minimize time and cost by not inspecting the same street more than once.



a) In this graph of the city, is it possible to begin at the garage (G) and inspect each street only once? Will you be back at the garage at the end of inspection?
b) Find a route that inspects all streets, repeats the least number of edges possible, and returns to the garage.

7. Construct the following digraphs.

a) $V = \{A, B, C, D, E\}$
$E = \{AB, CB, CE, DE, DA\}$

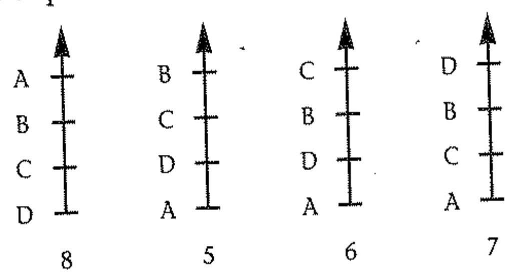
b) $V = \{W, X, Y, Z\}$
$E = \{WX, XZ, ZY, YW, XY, YX\}$

8. a. Construct a digraph for the following adjacency matrix.

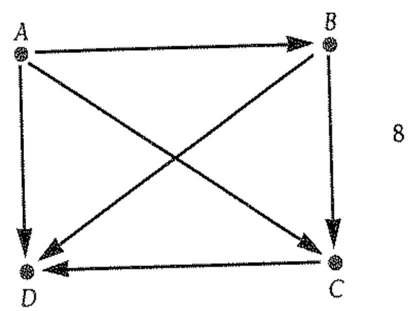
$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b. Is there symmetry along the main diagonal of the adjacency matrix? Explain why or why not.
c. Find the sum of the numbers in the second row. What does that total indicate?
d. Find the sum of the numbers in the second column. What does that total indicate?

9. Consider the set of preference schedules from Lesson 1.3:

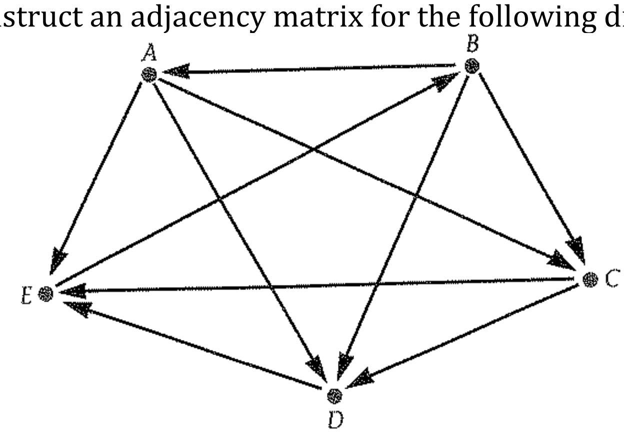


The first preference schedule could be represented by the following tournament.



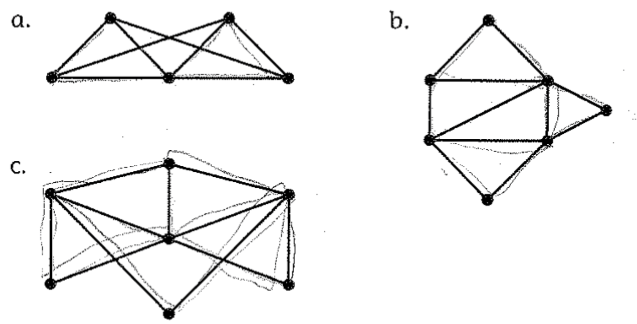
a. Construct tournaments for each of the three other preference schedules.
b. Construct a cumulative preference tournament that would show the overall results of the four individual preference schedules.
c. Is there a Condorcet winner in the election? (Recall from Lesson 1.3 that a Condorcet winner is one who is able to defeat each of the other choices in a one-on-one contest.)
d. Find a Hamiltonian path for the cumulative tournament. What does this path indicate?

10. Construct an adjacency matrix for the following digraph, and call it M.



Homework on Day 4 Material

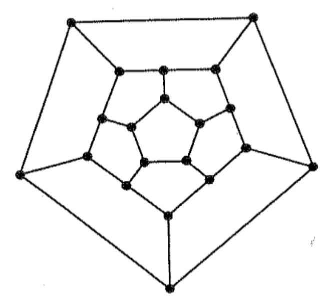
1. Which of the graphs have Hamiltonian circuits? Explain your reasoning.



2. a. Construct a graph that has both an Euler and a Hamiltonian circuit.

b. Construct a graph that has neither an Euler nor a Hamiltonian circuit.

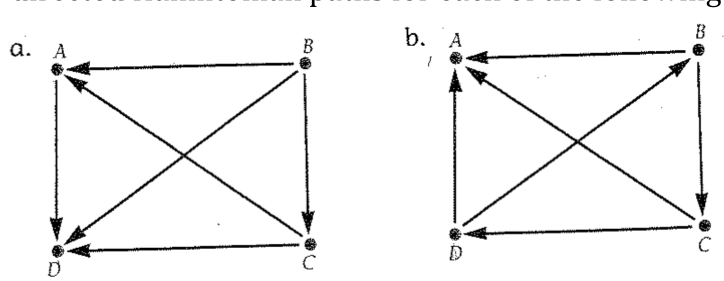
3. Hamilton's Icosian game was played on a wooden regular dodecahedron. Here is a planar representation of the graph.



a. Find a Hamiltonian circuit for the graph.
b. Is there only one Hamiltonian circuit for the graph?
c. Can the circuit begin at any of the vertices or only at some of them?

3. Draw a tournament with five players, in which player A defeats everyone, B defeats everyone but A, C is defeated by everyone, and D defeats E.

4. Find all the directed Hamiltonian paths for each of the following tournaments.

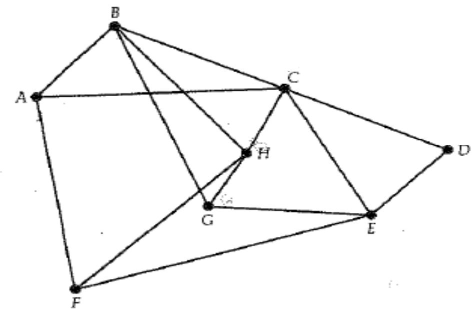


Homework Day 5

5. a. What is the chromatic number of K_2? K_3? K_4? K_5?

b. What can you say about the number of colors needed to color a complete graph? Explain your reasoning.

8. Mrs. Suzuki is planning to take her history class to the art museum. Following is a graph showing those students who are not compatible. Assuming that the seating capacity of the cars is not a problem, what is the minimum number of cars necessary to take the students to the museum?



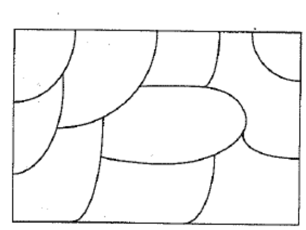
9. Mr. Butler bought six different types of fish. Some of the fish can live in the same aquarium, but others cannot. Guppies can live with Mollies, Swordtails can live with Guppies, Plecostomi can live with both Mollies and Guppies, Gold Rams can live only with Plecostomi, and Piranhas cannot live with any other fish. What is the minimum number of fish tanks needed to house the fish?

10. Following a list of chemicals and the chemicals with which each cannot be stored.

Chemicals	Cannot be Stored with:
1	2, 5, 7
2	1, 3, 5
3	2, 4
4	3, 7
5	1, 2, 6, 7
6	5
7	1, 4, 5

a. How many different storage facilities are necessary in order to keep all seven chemicals?

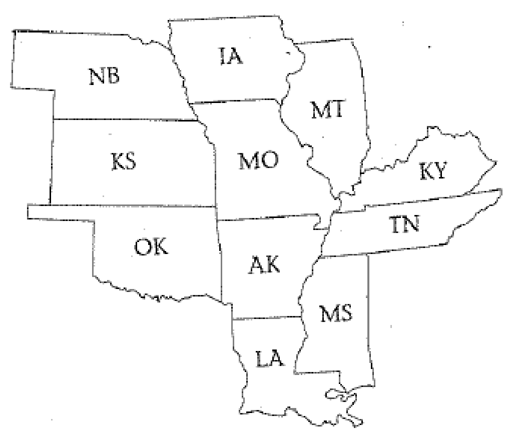
11. Color the following map to the right using only 3 colors.



12. Draw graphs to represent the following maps. Color the graphs. What is the minimum number of colors needed to color each map?

a.

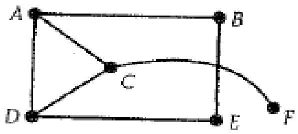
b.



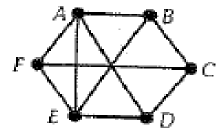
Homework Day 6

In Exercises 1-3, decide whether the graph is planar or nonplanar. If the graph is planar, redraw it without edge crossings.

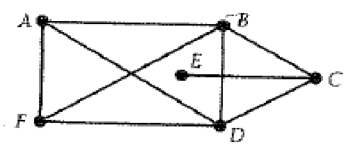
1.



2.



3.

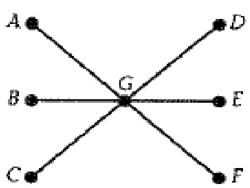


10. Construct the following bipartite graphs.

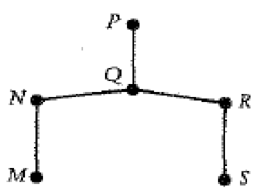
a. $K_{2,3}$ b. $K_{2,4}$

11. For each of the following bipartite graphs, list the two distinct sets into which the vertices can be divided. (Hint: use colors)

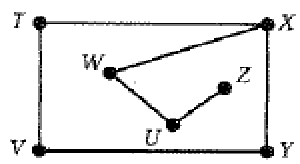
a.



b.

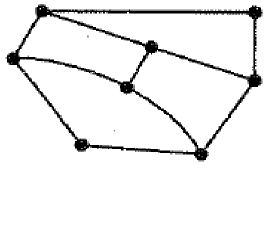


c.

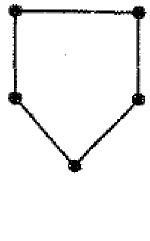


12. State whether the following graphs are bipartite. Explain why or why not. (Hint: use colors)

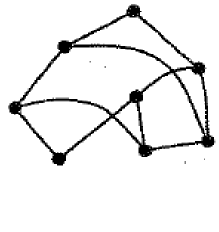
a.



b.



c.



14. How many edges are in a $K_{2,3}$ graph? A $K_{4,3}$? A $K_{m,n}$?

15. When does a bipartite graph $K_{m,n}$ have an Euler circuit?

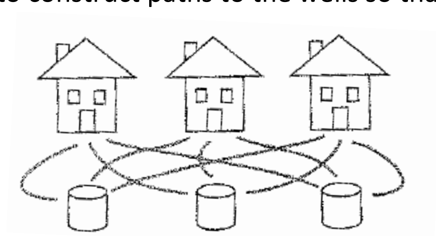
16. What is the chromatic number of a $K_{m,n}$ graph?

17. At Ms. Johnson's party, six men and five women walk into the dining room. If each man shakes hands with each woman, how many handshakes will occur? Represent this situation with a graph. What kind of a graph is it?

18. Describe a situation that can be represented by a bipartite graph that is not complete.

19. The following puzzle is often referred to as the *Wells and Houses problem* or the *Utilities problem*.

Three houses and three wells are built on a piece of land in an arid country. Because it seldom rains, the wells often run dry, and so each house must have access to each well. Unfortunately, the occupants of the three houses dislike one another and want to construct paths to the wells so that no two paths cross.



Draw a graph to illustrate this problem. Is it possible to satisfy the wishes of the feuding families? Explain why or why not.

Homework Day 7

Pg. 226 # 5-7

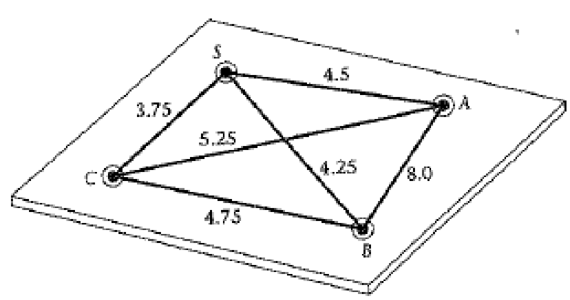
5. In a graph with 10 vertices, there are 9! possible Hamiltonian circuits exists if the beginning vertex is known.

   a. Assume that a computer can perform calculations at the rate of 1 million per second. About how long will it take the computer to check 9! possibilities? What if the graph had 15 vertices (14! possible circuits)?

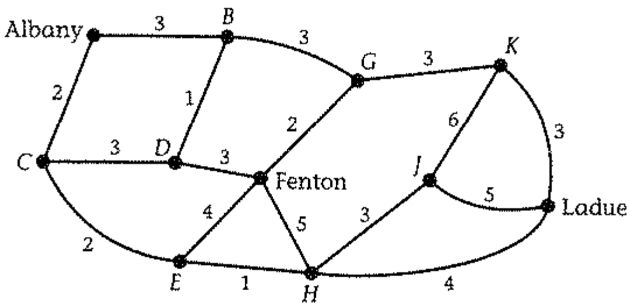
   b. According to October 29, 1998 news reports, a computer now exists that can do 1 trillion computations per second! How long will it take this new computer to check a graph with 9! circuits? With 14! possible circuits?

6. Give two examples of a situation in which a solution to the traveling salesperson problem would be beneficial.

7. The following figure shows a circuit board and the distances in millimeters between holes that must be drilled by a drilling machine. Since it is advantageous in terms of time to minimize the distance traveled, find the shortest possible circuit for the machine to travel and the total distance for that circuit. (Assume the machine has to begin and end at point *S*.)



5.

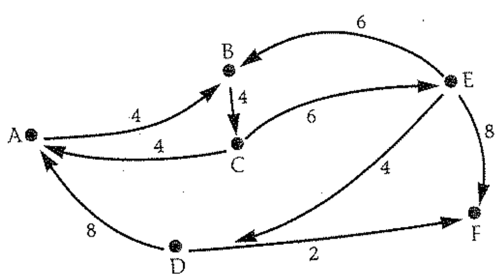


a. Use the shortest path algorithm to find the shortest route from Albany to Ladue in the preceding graph.

b. Assume it is necessary to travel from Albany to Fenton to deliver a package and then to continue from there to Ladue. Find the shortest route for this trip. Explain why the solution to this question might be different than the shortest route from Albany to Ladue.

6. In the shortest path algorithm, each time you examine the uncircled vertices that are adjacent to the circled ones, you have to recalculate the lengths of the paths from the starting vertex. Explain how the efficiency of the algorithm might be improved by modifying it to avoid such recalculation.

7. The shortest path algorithm can be applied to digraphs if slight modifications are made. Make the appropriate changes, and try your revised algorithm on the following digraph to find the shortest route from A to F.



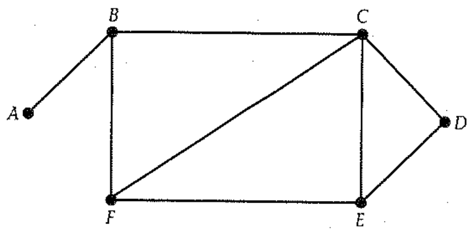
8. Mail Packages, Inc., ships from certain cities in the US to others. A table of the company's shipping costs follows.

	Albany	Biloxi	Center	Denver	Evert	Fargo	Gale
Albany	-	7	-	-	4	-	-
Biloxi	-	-	-	-	-	-	6
Center	2	-	-	-	2	-	-
Denver	-	-	1	-	-	-	-
Evert	-	-	-	-	-	-	4
Fargo	-	-	-	-	3	-	2
Gale	1	6	-	-	-	1	-

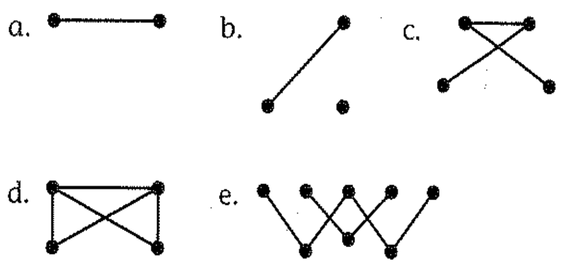
Since a package can't be shipped directly from Denver to Biloxi, construct a digraph to represent the cost table and apply the shortest path algorithm to find the least charge for shipping the package.

Homework Day 8

1. Examine the following graph for cycles. List as many as you can.

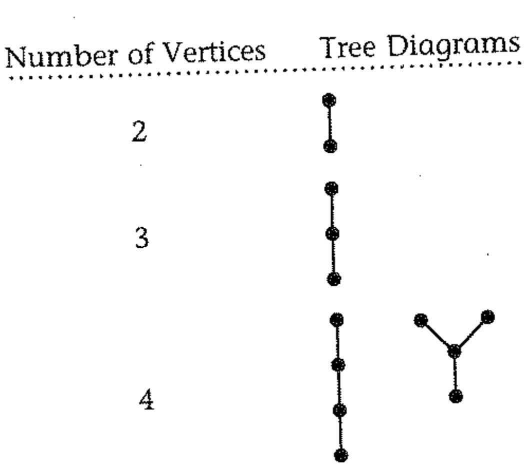


2. Determine whether the following graphs are trees. If the figure is not a tree, explain why.



a. ____
b. ____
c. ____
d. ____
e. ____

3. There is only one way to draw a tree with two vertices and only one way to draw a tree with three vertices, but there are two distinct trees that can be formed from four vertices.
Draw all the trees that are possible for five vertices and for six vertices.



Number of Vertices	Tree Diagrams
5	
6	

4. Complete the following table for trees with the indicated numbers of vertices.

Number of Vertices	**Number of Edges**
1	0
2	1
3	
4	
n	

   a. How many edges does a tree with 19 vertices have?

   b. How many vertices does a tree with 15 edges have?

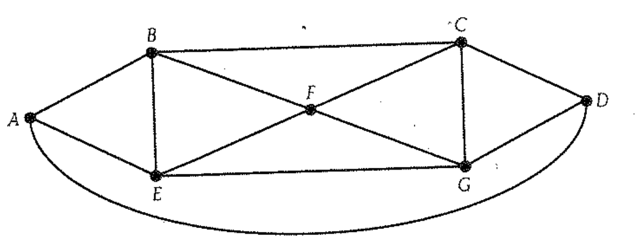
   c. What is the relationship between the number of vertices of a tree and the number of edges?

5. What happens to a tree if an edge is removed from it?

6. Draw a tree with six vertices that has exactly three vertices of degree 1.

7. In a hierarchical tree, it is natural to speak of the root of the tree. A tree is rooted when all of the edges are directed away from the chosen vertex (root). Draw a family tree for your family beginning with one of your grandfathers as the root of the tree. What do the leaves of your tree have in common?

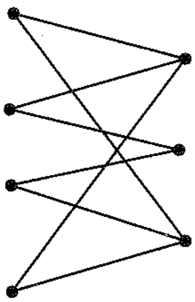
8. For the following graph, find two different **subgraphs** that are trees.



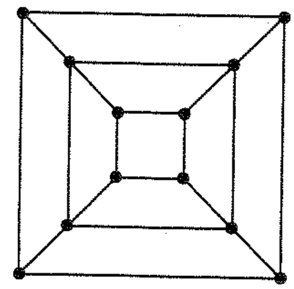
p. 249 # 1-5, 10-15

In Exercises 1-4, find a spanning tree for each graph if one exists.

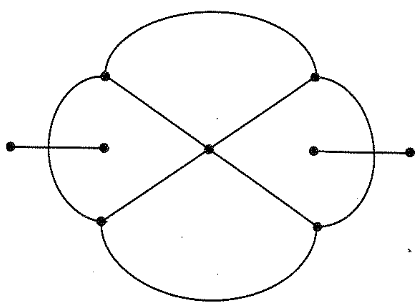
1.



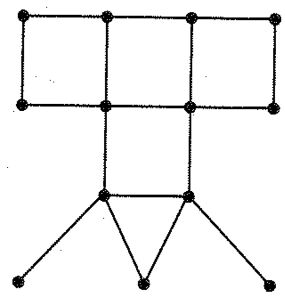
2.



3.



4.

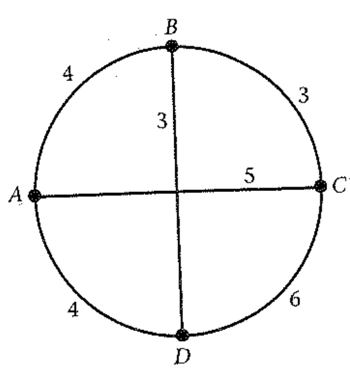


5. Draw a spanning tree for a K_5 graph.

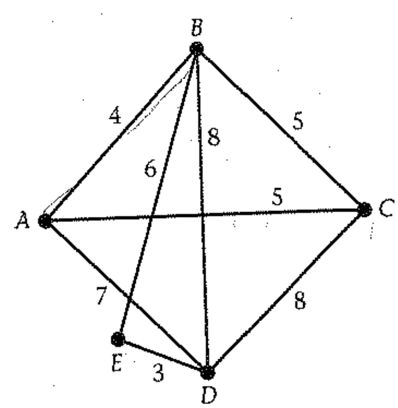
Use Kruskal's algorithm to find a minimum spanning tree for the graphs in 10-12.

What is the minimal weight in each case? Show all work.

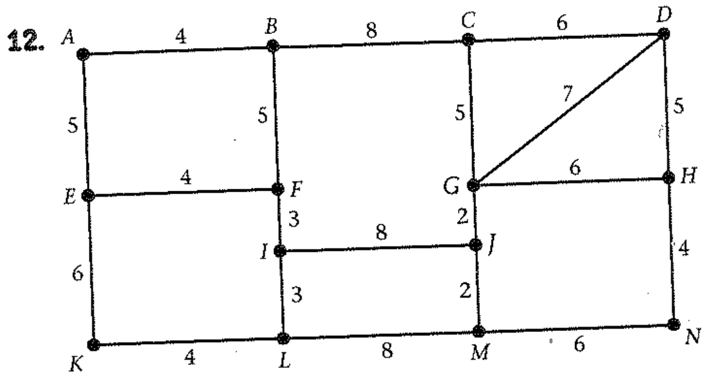
10.



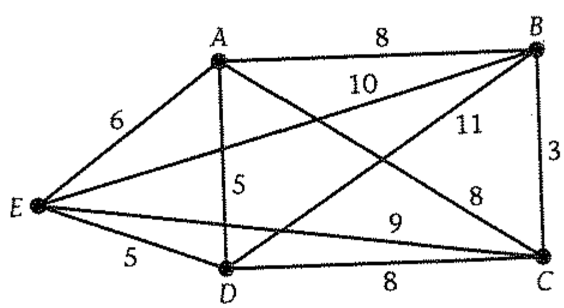
11.



12. Find the minimum spanning tree. What is the minimal weight?
Show your work by **creating a table.**



13. The computers in each of the offices at Pattonville High School need to be linked by cable. The following map shows the cost of each link in hundreds of dollars. What is the minimum cost of linking the five offices? Show your work by creating a table.



14. Suppose that the cable in Exercise 13 was installed by a disreputable firm that used only the most expensive links. What would be the maximum cost of the four links?

Homework Day 9

In Exercises 8-11, represent the expression as a binary expression tree.

8. (2 – 5) * (4 + 7)

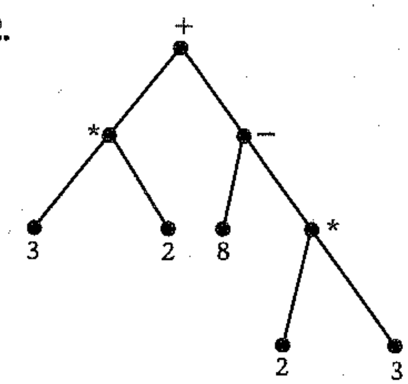
9. (2 + 3) * 4

10. 2 + 3 * 4 – 6/2

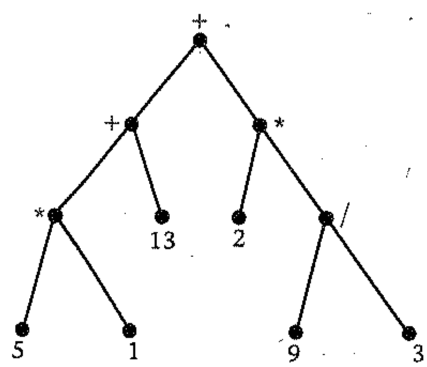
11. A * B + (C – D/E)

In Exercisees 12-14, find the postorder listings for each binary tree.

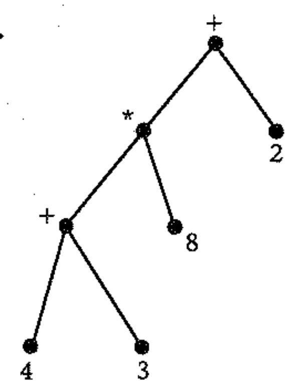
12.



13.



14.



15. Evaluate the following reverse Polish notations.

a. 6 2 – 7 * 3 2 + +

b. 6 5 4 3 2 – + / +

c. 1 2 + 4 3 –+ 6 2 / 2 + +

d. 4 3 + 8 2 – + 4 + 3 –

16. Give the reverse Polish notation for each of the following expressions.

a. 2 + 3 * 6 – (4 + 1)

b. (5 – 3) * 2 + (7 – 6/2)