ICM Matrices and Game Theory **Name: ___________________________**

Day 1 Practice: Do you know matrices?

Let $A=\begin{pmatrix} -2 & 2 & 3y \\ 1 & -x & 0 \\ 0 & 1 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 0 & -3x & 7 \\ y & 1 & 6 \\ 0 & 2 & -4 \end{pmatrix}$, $C=\begin{pmatrix} -3 & 8 \\ 9x & -12 \end{pmatrix}$, $D=\begin{pmatrix} -5 & 0 \\ 2x & 7 \end{pmatrix}$

Evaluate. Remember to show work for credit! If the solution is undefined, explain why.

1. A + B
2. 2C – D
3. B + C
4. Find AB.
5. Find BA.
6. Find CD.
7. Find DC.
8. Find AC.
9. Is matrix addition commutative or associative or both? Prove your answer.
10. Is matrix multiplication commutative or associative or both? Prove your answer.

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Matrix Applications

Remember to show your work for credit!

1. Two softball teams submit equipment lists for the season.
   Women's Team: 12 bats, 45 balls, 15 uniforms
   Men's Team: 15 bats, 38 balls 17 uniforms

   Each bat costs $21, each ball costs $4, and each uniform costs $30.

   (A) Set up and label two matrices.

   (B) Use matrix multiplication to find the total cost of equipment for each team.

2. Matrix S gives the number of three types of cars sold in March by two car dealers, and matrix P gives the profit for each type of car sold.

$$S = \begin{array}{c} \text{Dealer} \\ \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 12 & 10 \\ 40 & 15 \\ 17 & 42 \end{bmatrix} \end{array} \qquad P = \begin{bmatrix} \$400 & \$650 & \$900 \end{bmatrix}$$

   (A) Which matrix is defined, SP or PS? Explain.

   (B) Find the defined matrix from part A.
   Interpret its elements.

3. A Chicago Company wants to send some of its key personnel to a convention in London. In the company's Research and Development Division, five people plan to fly first class, three people plan to fly business class, and two people plan to fly coach class. In the Sales Division, four people plan to fly business class, and eight people coach class.

   (A) Display this information in a 2 x 3 travel matrix T.
   Label the matrix.

   Round-trip prices for four different airlines are as follows: Airline A charges $1,280 for first class, $922 for business class, and $676 for coach. Airline B charges $1,400 for first class, $1,024 for business class, and $728 for coach. Airline C charges $1,320 for first class, $905 for business class, and $654 for coach. Airline D charges $1,450 for first class, $1,050 for business class, and $734 for coach.

   (B) Display this information in a price matrix P that can be multiplied with matrix T to give the travel costs for each company division per airline.

   (C) Find the product.

   (D) How much will it cost to fly the Sales Division on Airline D?

   (E) Which airline will cost the Research and Development Division the least?

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Practice 3.4 - 3.5

The following data is for a certain SPECIES of hamster.

Age(months)	**0 - 4**	**4 - 8**	**8 - 12**	**12 - 16**	**16 - 20**	**20 - 24**
Birthrate	0	0.6	1.3	1.1	0.5	0
Survival Rate	0.7	0.9	0.8	0.7	0.5	0
Initial Population	22	10	12	8	8	0

1. What is the expected life span of these hamsters?

2. Create the Leslie Matrix for this population.

3. Find the population distribution after 2 cycles (8 months).

4. Find the total population after 3 cycles (12 months).

5. Find the newborn population after 6 cycles (24 months).

Suppose an animal population begins reproducing after two 3-year cycles, and has the characteristics described in the table below.

6.

Age Groups (years)	**0-3**	**3-6**	**6-9**	**9-12**	**12-15**	**15-18**	**18-21**
Survival rate	0.5	0.6	0.8	0.9	0.9	0.4	0
Birth rate	0	0	1.2	0.8	0.8	0.3	0
Initial Population Distribution	12	20	35	40	52	28	0

Construct the Leslie Matrix for this animal.

7. Using the initial female population distribution from above, find the population age distribution after 12 years.

8. What is the total female population after 12 years?

9. What is the long-term growth rate for this population?

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HOMEWORK: SECTION 3.4 p. 145 #2

2. Suppose that a species of deer has the following birth and survival rates.

Age (years)	**Birthrate**	**Survival Rate**
0-2	0	0.6
2-4	0.8	0.8
4-6	1.7	0.9
6-8	1.7	0.9
8-10	0.8	0.7
10-12	0.4	0

a) Given that the initial population for this species is 148 deer with the following distribution, find the number of newborn female deer after 2 years (1 cycle).

Age (years)	0-2	2-4	4-6	6-8	8-10	10-12
Number	50	30	24	24	12	8

b) Arrange the initial population distribution in a row matrix and the birth rates in a column matrix. Interpret this result. Multiply the row matrix times the column matrix. Interpret the result.

c) Calculate the number of deer that survive in each age group after 2 years and move up to the next age group.

d) Explore the possibility of multiplying the initial population distribution in a row matrix times some column matrix to find the number of deer after 2 years that move from the 2-4 to the 4-6 group.

5. Consider the deer population from #2 in Section 3.4 that is shown below.
Suppose that a species of deer has the following birth and survival rates.

Age (years)	**Birthrate**	**Survival Rate**
0-2	0	0.6
2-4	0.8	0.8
4-6	1.7	0.9
6-8	1.7	0.9
8-10	0.8	0.7
10-12	0.4	0

a) Construct the Leslie matrix for this animal.

b) Given that $P_0 = \begin{bmatrix} 50 & 30 & 24 & 24 & 12 & 8 \end{bmatrix}$, find the long-term growth rate.

c) Suppose the natural range for this animal can sustain a herd that contains a maximum of 1,250 females. How long before this herd size is reached?

d) Once the long-term growth rate of the deer population is reached, how might the population of the herd be kept constant?

A certain species of moth lives only 4 weeks. 80% survive from the first week to the second week, 60% survive from the second week to the third week, and 30% survive to the fourth week of life. On average, each female moth that survives into the fourth week produces 35 new female moths. Only those moths that survive to the fourth week of life produce offspring.

10. Construct a Leslie Matrix representing the life cycle of these moths. The initial population of moths is represented by this matrix:

$$P_0 = [\ 0 \quad 0 \quad 150 \quad 0\ ]$$

11. Find the population distribution after 6 cycles, 7 cycles, 8 cycles, 9 cycles, and 10 cycles.

12. What is the newborn population after 6 cycles?

13. What is the total population after 12 cycles?

14. What is the long-term growth rate of this species of moth?

Practice: SECTION 7.3 Markov and Transition Matrices

1. a. Find the distribution of students eating or not eating in the cafeteria each day for the first week of school using the initial distribution $D_0 = \begin{bmatrix} .75 & .25 \end{bmatrix}$ and the transition matrix T of this lesson.

   b. Find the distribution of students eating and not eating in the cafeteria after 2 weeks (10 school days) have passed. Repeat for 3 weeks (15 days).

   c. What would your report to the director of food services be, based on your computations in parts a and b?

   d. Choose any other initial distribution of students and repeat parts a and b.

   e. Compare the results of parts b and d. Does the initial distribution appear to make a difference in the long run?

   f. Calculate the 15^{th} power of matrix T. Compare the entries in T^{15} to the distribution after the 15^{th} day.

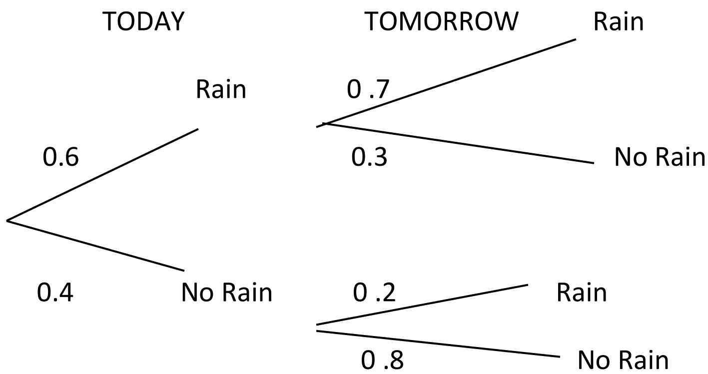
2. Suppose the entire student body eats in the cafeteria on the first day of school. The initial distribution in this case would be D0 = $\begin{bmatrix} 1 & 0 \end{bmatrix}$. Repeat parts a and b of Exercise 1 for this distribution. After several weeks, what percentage of students will be eating in the cafeteria?

3. Which of the matrices below could be Markov transition matrices? For the matrices that could not be transition matrices, explain why not.

$a. \begin{pmatrix} .7 & .3 \\ .6 & .6 \end{pmatrix}$ $b. \begin{pmatrix} .1 & .4 & .5 \\ .2 & .6 & .2 \\ & & \end{pmatrix}$ $c. \begin{pmatrix} 1.2 & -4 \\ 1 & 0 \end{pmatrix}$

$d. \begin{pmatrix} .6 & .3 & .1 \\ .3 & .3 & .3 \\ & & \end{pmatrix}$ $e. \begin{pmatrix} .75 & .25 \\ 1 & 0 \end{pmatrix}$ $f. \begin{pmatrix} .45 & .55 \\ .33 & .66 \end{pmatrix}$

4. There is 60% chance of rain today. It is known that tomorrow's weather depends on today's according to the probabilities shown in the following tree diagram.



a. What is the probability it will rain tomorrow if it rains today?
b. What is the probability it will rain tomorrow if it doesn't rain today?
c. Write an initial-state matrix that represents the weather forecast for today.
d. Write a transition matrix that represents the transition probabilities shown in the tree diagram.
e. Calculate the forecast for 1 week (7 days) from now.
f. In the long run, for what percentage of days will it rain?

5. A taxi company has divided the city into three districts—Westmarket, Oldmarket, and Eastmarket. By keeping track of pickups and deliveries, the company found that of the fares picked up in the Westmarket district, only 10% are dropped off in that district, 50% are taken to the Oldmarket district, and 40% go to the Eastmarket district. OF the fares picked up in the Oldmarket district, 20% are taken to the Westmarket, 30% stay in the Oldmarket, and 50% are dropped off in the Eastmarket. Of the fares picked up in the Eastmarket district, 30% are delivered to each of the Westmarket and Oldmarket districts, while 40% stay in the Eastmarket district.

a. Draw a tree diagram to represent the probabilities in this scenario.

b. Construct a transition matrix for these data.

c. Write an initial-state matrix for a taxi that starts off by picking up a fare in the Oldmarket district. What is the probability that it will end up in the Oldmarket district after three additional fares?

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HOMEWORK: SECTION 7.3 p. 366 #7-8, 11-13

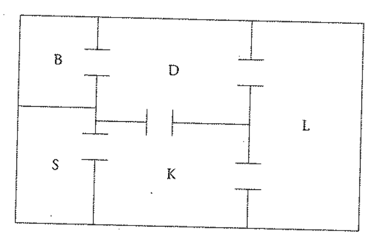
7. Emily, Jon, and Gretchen are tossing a football around. Emily always tosses to Jon, and Jon always tosses to Gretchen, but Gretchen is equally like to toss the ball to either Emily or Jon.

a) Draw a tree diagram to represent the probabilities in this scenario.

b) Represent this information as the transition matrix of a Markov chain.

c) What is the probability that Emily will have the ball after three tosses if she was the first one to throw it to one of the others?

8. Jim agreed to care for Emily's cat, Ellington, for the weekend. On Friday night, Ellington prowled the first floor of Jim's house, randomly moving from room to room, not staying in one room for more than a few minutes. The following floor plan shows the location of the rooms and doorways in Ellington's range. The letters on the floor plan represent Living Room, Dining Room, Kitchen, Bathroom, and Study.



Each of Ellington's movements can be interpreted as a transition in a Markov chain in which a state is identified with the room he is in. The first row of the transition matrix is

$$\begin{array}{cc} & \begin{array}{ccccc} L & D & K & S & B \end{array} \\ L & \left[\begin{array}{ccccc} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array}\right] \end{array}.$$

a) Construct the complete transition matrix for this situation.

b) If Ellington starts off in the living room, what is the probability that he will be in the study after two transitions? After three transitions?

c) After a large number of transitions, what is the probability that Ellington will be in the bathroom?

d) In the long run, what percentage of the time will Ellington spend in the kitchen or the dining room?

12. A group of researchers are studying the effect of a potent flu vaccine in healthy (well) and infirm (ill) rats. When the rats are injected with the vaccine, three things may occur. The rat may have no reaction where its health status does not change. The rat may have a mild reaction and become ill, or the rat may have a severe reaction and die. The probabilities of each of these reactions are shown in the following matrix.

$$\begin{array}{ccc} \textit{Well} & \textit{Ill} & \textit{Dead} \end{array}$$
$$\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Write an initial-state vector for a healthy rat who is injected with the vaccine.

b) In this study, the scientists check the status of the rats on a daily basis. Use the transition matrix to predict the health of the rat in part a after 4 days.

c) Use the transition matrix to predict the rat's health in the long run.

13. A hospital categorizes its patients as well (in which case they are discharged), good, critical, and deceased. Data show that the hospital's patients move from one category to another according to the probabilities shown in the transition matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & .3 & .2 & 0 \\ 0 & .3 & .6 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a) Write an initial-state matrix for a patient who enters the hospital in critical condition.

b) If patients are reclassified daily, predict the patient's future after one week in the hospital.

c) Predict the future of any patient in the long run.

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HOMEWORK: SECTION 7.4 p. 38 #6-8

6. Two major discount companies, Salemart and Bestdeal, are planning to locate stores in Nebraska. If Salemart locates in city A and Bestdeal in city B, then Salemart can expect an annual profit of $50,000 more than Bestdeal's annual profit. If both locate in city A, they expect equal profits. If Salemart locates in City B and Bestdeal in City A, then Bestdeal's profits will exceed Salemart's by $25,000. If both companies locate in city B, then Salemart's profits will exceed Bestdeal's by $10,000. What are the best strategies in this situation and what is the saddle point of the game?

7. Jon and Gretchen each have three dimes. They both hold either one, two, or three coins in a clenched fist and open their fists together. If they both are holding the same number of coins, Jon will take the coins that Gretchen is holding. If they are holding different numbers of coins, then Gretchen will take the coins that Jon is holding.

   a. Write the payoff matrix from Jon's point of view.

   b. Does this game have a saddle point? If so, what are the strategies for Jon and Gretchen?

8. Mike is going to see his girlfriend, Nancy, after track practice, when he suddenly remembers that today may be a special anniversary for Nancy and him, and he always brings her a single red rose on this occasion. Be he's not sure. Maybe the anniversary is next week. What should he do? If it is their anniversary and he doesn't bring a rose, then he'll be in bad trouble. On a scale from 0 to 10, he's score a – 10. If he doesn't bring a rose and it isn't their anniversary, Nancy won't know anything about this frustration, and he'll score a 0. If he brings a rose and it is not their anniversary, then Nancy will be suspicious that something funny is going on but he'll score about a 2. If it is their special anniversary and he brings a rose, then Nancy will be expecting I and he'll score a 5. Write a payoff matrix for this situation. What is Mike's best strategy?

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HOMEWORK: SECTION 7.5 #8-10

8. A group of parents in a small town in the Midwest are in an uproar about a new social studies program that the school district has adopted. They are seeking to have the program removed from the curriculum. A second group of parents believe the new program is a solid choice and are organizing in favor of keeping it. In order to bring the issue before the voters in the town, the opposing group must collect 400 supporting signatures from registered voters. Both sides are campaigning vigorously by making telephone calls, send emails and going door to door to contact voters. The local newspaper has estimated the number of signatures. What are the best strategies for both groups of parents? If both follow their best strategies, can the opposing group expect to gather enough signatures to get the issue to ballot?
(Hint: Use the concept of dominance to eliminate a row and column.)

$$\begin{array}{ccc} & & \text{Group in Favor} \\ & & \begin{array}{ccc} \textit{Phone} & \textit{Email} & \textit{Door} \end{array} \\ \text{Group Against} & \begin{array}{c} \textit{Phone} \\ \textit{Email} \\ \textit{Door} \end{array} & \begin{bmatrix} 150 & 75 & 100 \\ 350 & 300 & 200 \\ 500 & 100 & 400 \end{bmatrix} \end{array}$$

10. In a campaign for student council president at Northeast High, the top two candidates, Betty and Bob, are each making two promises about what they will do if they are elected. The payoff matrix in terms of the number of votes Betty will gain follows. What is the best strategy for each candidate and what is Betty's expectation?

$$\begin{array}{ccc} & & \textit{Bob} \\ & & \begin{array}{cc} A & B \end{array} \\ \textit{Betty} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 200 & 100 \\ 50 & 180 \end{bmatrix} \end{array}$$

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Practice 7.3-7.5

1. Billy Joe and Bobbie Sue are arrested by the police. Together they committed a serious crime but the police have insufficient evidence for a conviction. The police separate the prisoners and visit each of them separately to offer them a deal.
   Here's the deal.
   If one testifies against the other and the other remains silent, then the testifier will go free and the other will receive a full 15-year prison sentence. If both remain silent, the police will not be able to get a conviction for the serious crime. Both will be sentenced to 8-months in jail for a minor charge. If both choose to testify, a more lenient sentence will be given and both prisoners will receive a 5-year prison sentence. Each prisoner must choose to testify or remain silent.

   Make a payoff matrix for this game. Use Billy Joe's point of view, so he will be the row player. What should each prisoner do?

2. School board and Teacher Education Association representatives are meeting to negotiate a contract. Each side can either threaten (reduction in staff or strike), refuse to negotiate, or negotiate willingly. Each side decides its strategy prior to coming to the negotiating table. The following payoff matrix gives the percentage pay increases for the teachers that would result from each combination of strategies. Find the best strategies for each side.

		School Board		
		Threaten	Refuse	Negotiate
	Threaten	5	4	3
Teachers	Refuse	3	0	2
	Negotiate	4	3	2

3. Katy has a dance class that meets on Tuesday, Thursday, and Friday. She decides on any day to attend class based on whether she has attended the previous class. If she attends a class, she will attend the next class with a probability of 0.78, but if she misses a class she will attend the next class with probability 0.84.

   a. Set up a transition matrix for the changes.

   b. If Katy attends class on Tuesday, find the probability that she will skip class on Thursday of the next week.

   c. If this trend continues, how likely is Katy to attend the dance class?

4. In a certain culture a person's level of education depends on that of his or her parents. If the parents' highest level of education is college, 54% of their children will achieve a college education, 33% will complete a high school education, and 13% will complete only a grade school education. If the parents' highest level of education is high school, 36% of their children will achieve a college education, 52% will complete a high school education, and 12% will complete only a grade school education. If the parents' highest level of education is grade school, 11% of their children will achieve a college education, 53% will complete a high school education, and 36% will complete only a grade school education.
   a. Represent this situation as a transition matrix of a Markov chain process. Label the rows and columns.

   b. What is the probability that the *great-grandchild* of a high school graduate finishes college?

5. The matrix to the right represents a payoff matrix for a game. $\begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix}$
   a. Calculate the best strategy for the row player.
   b. Calculate the best strategy for the column player.
   c. If both players play their best strategy what is the expected payoff for the row player?

6. For your cross-town subway ride, you can either buy a ticket for \$4.75 or ride illegally without a ticket. Riding illegally is free if the transit conductor does not catch you, but there is a \$38 fine if you do get caught. However, you resent buying a ticket when the transit conductor is not around, and you value this outcome as costing you \$6 (\$4.75 for the ticket and another \$1.25 for the inconvenience of getting the ticket). The transit conductor considers this a zero-sum game so that your losses are viewed as gains for her.

   C = Caught
   NC = Not Caught
   T = Have Ticket
   NT = No Ticket

   a. Fill in the following payoff matrix to describe this situation

conductor

$$\begin{array}{ccc} & & \begin{array}{cc} \text{C} & \text{NC} \end{array} \\ \textit{you} & \begin{array}{c} \text{T} \\ \text{NT} \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

   b. Determine the best strategies for you and the transit conductor. If the game is strictly determined, find the saddle point. If not, find the expected payoff to you when both you and the transit conductor are using your best strategies.