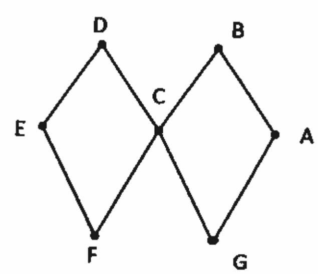
Key with some work

Unit 7 Day 4
Section 4.5
& Practice 4.1-4.5

Hamiltonian Circuits and Paths

1

Warm Up ~ Day 4



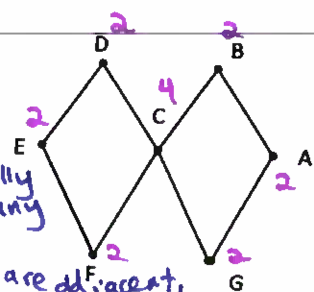
- Is the following graph...

1) Connected?
2) Complete?
3) An Euler Circuit? If so, write the circuit. If not, explain why not.
4) An Euler Path? If so, write the path. If not, explain why not.

5) What is the degree of vertex C?

2

Warm Up ~ Day 4

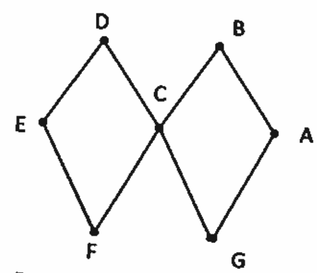


- Is the following graph...

1) Connected? Yes. You can eventually get between any 2 vertices.
2) Complete? No. Not all vertices are adjacent.
3) An Euler Circuit? If so, write the circuit. If not, explain why not. Yes because all degrees are even. Example: EDCBAGCFE or EDCGABCFE
4) An Euler Path? If so, write the path. If not, explain why not. No. Need exactly 2 odd degree vertices for an Euler path.
5) What is the degree of vertex C?
   4 (because 4 edges connected to C or because 4 vertices adjacent to C)

2

Warm Up ~ Day 4
ANSWERS



- Is the following graph...

1) Connected? Yes!
2) Complete? No!
3) An Euler Circuit? If so, write the circuit. If not, explain why not. Yes! E,D,C,B,A,G,C,F,E
4) An Euler Path? If so, write the path. If not, explain why not. No, its an Euler circuit! They are mutually exclusive!
5) What is the degree of vertex C?
   deg(C)=4

3

Section 4.3 Answers

4

Packet p. 3

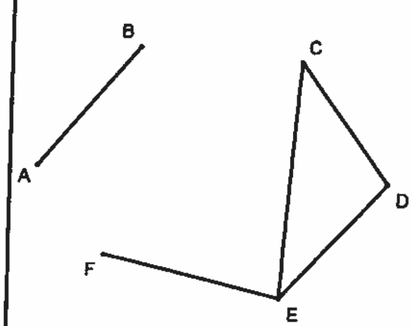
3. Draw a graph with vertices = {A, B, C, D, E, F} and edges = {AB, CD, DE, EC, EF}.

a. Name two vertices that are not adjacent.

b. F, E, C is one possible path from F to C. This path has a length of 2, since two edges were traveled to get from F to C. Name a path from F to C with a length of 3.

c. Is this graph connected? Explain why or why not?

d. Is this graph complete? Explain why or why not?



a. Sample Answer: F, D

b. F, E, D, C

c. No. There is no path from A to F

d. No. To be complete, every vertex must be connected to every other vertex.

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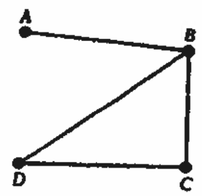
Packet p. 3

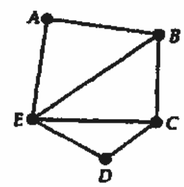
5. Construct a graph for each adjacency matrix. Label the vertices $A, B, C, \ldots$.

a. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

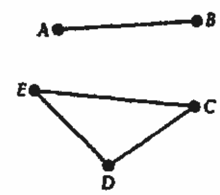
a.



b.



c.



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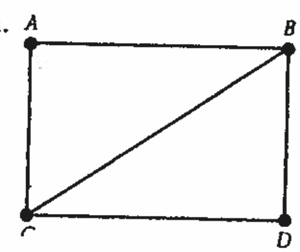
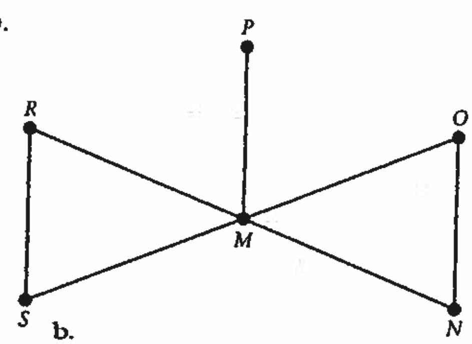
Packet p. 3

6. Create an adjacency matrix for each of the following graphs:

a.



b.



a.

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b.

$$\begin{array}{c|c} & \begin{matrix} M & N & O & P & R & S \end{matrix} \\ \begin{matrix} M \\ N \\ O \\ P \\ R \\ S \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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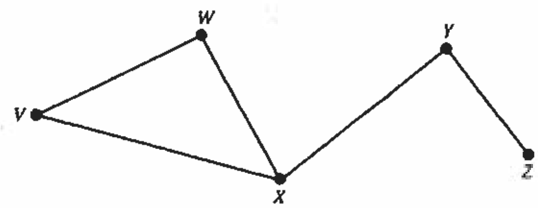
Section 4.3 Exercise #7 p.169

7. Give the adjacency matrix for the following graph.



a. What do you notice about the main diagonal of the matrix?
b. Does your matrix possess symmetry? If so, where?
c. If an adjacency matrix has a 1 on the main diagonal, what would that indicate? What would a 2 in row 2, column 1 indicate?

$$\begin{array}{c} \begin{array}{ccccc} V & W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a. All zeros
b. Yes, along the main diagonal
c. A vertex adjacent to itself.
Two edges between two vertices.

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Packet p. 3-4

7. The number of vertices that are adjacent to a given vertex; the degree of each vertex.

8. deg(V)=3, deg(W)=4, deg(X)=2, deg(Y)=2, deg(Z)=1

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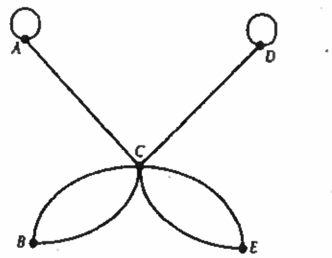
Packet p. 4

9. An edge that connects a vertex to itself is called a **loop**. If a graph contains a loop or **multiple edges** (more than one edge between two vertices), the graph is known as a **multigraph**. When finding the degree of a vertex on which there is a loop, the loop is counted twice. For example, deg(A) = 3.



* b.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

a. Find the degree of vertices B, C, D, and E.

* b. Give the adjacency matrix for the above multigraph.

c. Compare an adjacency matrix for a graph and one for a multigraph. Without seeing the graph, can you tell which belongs to the graph and which belongs to the multigraph? Explain how you know.

a. deg(B)=2, deg(C)=6, deg(D)=3, deg(E)=2

c. Yes, you can tell the difference. A graph will have 0s on the main diagonal and 1s and 0s elsewhere. A multigraph is characterized by numbers greater than 1 in the graph (indicating multiple edges) and/or numbers other than 0s on the main diagonal (indicating loops).

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Packet p. 4-5

10. a) Euler Circuit – all even degree vertices
    b) Neither – multiple odd degree vertices
    c) Euler Path – 2 odd degree vertices, rest even
    d) Euler Circuit – all even degree vertices

11. Answers may vary – one possible:
    e,d,f,h,d,c,h,b,c,g,a,h,g,f,e

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Packet p. 5

12. Only complete graphs with odd number of vertices will have Euler circuits: K_3 K_5 K_7 ...K_{2n-1}

9. a) Yes b) No c) Yes

10. a) No, not the same number of indegrees as outdegrees.
    b) Yes! b, e, f, g, c, b, a, c, d, f
    Start at mismatched in and out degrees.

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Tonight's Homework

- Finish Review Packet p. 6-8
- Packet p. 9

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Notes Day 4
Hamiltonian Circuits and Paths
Section 4.5

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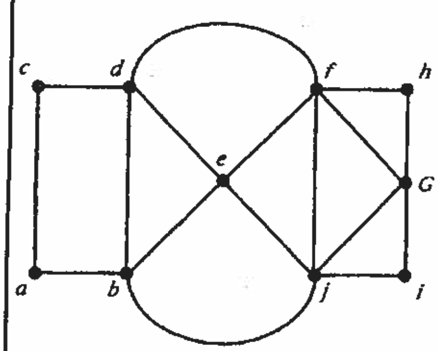
The last section was about Euler Circuits that visit each EDGE only once. This section is about visiting each VERTEX only once.



Suppose you are a city inspector, but instead of inspecting the streets, you must inspect the fire hydrants at each street intersection.

Can you start at the Garage (G), visit each intersection only once and return to the Garage? Try it.

A graph that visits each vertex only once is known as a **HAMILTONIAN PATH**

If that Hamilton path can end at the starting vertex, it is called a **HAMILTONIAN CIRCUIT**

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Euler Circuit:
- visit each edge once + return to starting vertex at end

→ Ex: mailman at entrance of neighborhood and hit each side of road once to deliver mail then return to entrance of neighbor-hood

Sir William Rowan Hamilton (1805-1865)

- How old was he when he died? 60 years old
- Irish mathematician, appointed astronomer and knighted at age 30.
- Born in Dublin, Ireland and the 4th of 9 children!
- He carries the title of discovering Algebra.

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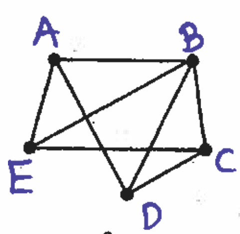
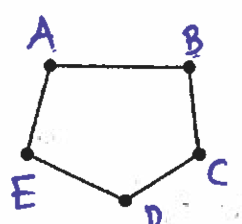
Hamiltonian Circuits & Paths

Try to find a Hamiltonian Circuit for each of the graphs.



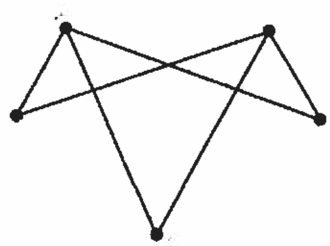
a.

Yes, Example BCDAEB is a Hamiltonian circuit



b.

Yes, Example ABCDEA is a Hamiltonian circuit



c.

No Hamiltonian circuit

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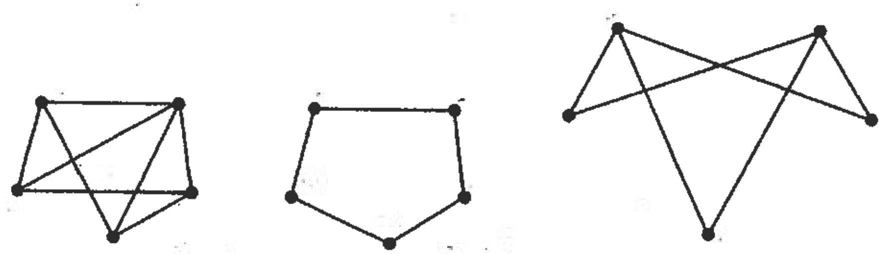
Hamiltonian Circuits & Paths ANSWERS

Try to find a Hamiltonian Circuit for each of the graphs.



a.	b.	c.
Yes it does have a Hamilton Circuit	Yes, it has a Hamilton Circuit	Does NOT have a Hamilton Circuit

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Hamiltonian Circuits & Paths

A simple test for determining whether a graph has a Hamiltonian circuit has not been found. It may be impossible.

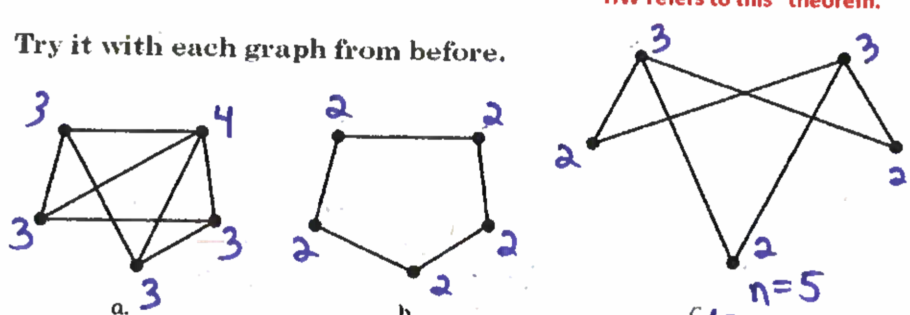
There is a test to guarantee the existence of a Hamiltonian circuit. If the graph fails the test, however, there still may be a Hamiltonian circuit.

> If a connected graph has n vertices, where $n > 2$ and each vertex has degree of a least $n/2$, then the graph has a Hamiltonian circuit.

HW refers to this "theorem."

n = # of vertices
① over 2 vertices
② every vertex has degree at least $\frac{n}{2}$
⇒ has Hamiltonian circuit

Try it with each graph from before.



$n=5$
✓ ① $n>2$

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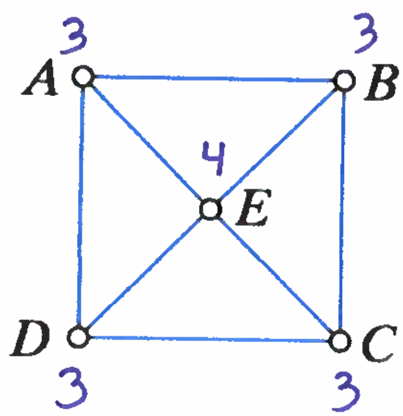
$n=5$
① $n>2$ ✓
② all vertices ✓ degree $> \frac{n}{2}$ (2.5 here)
Yes Hamiltonian circuit

$n=5$
✓ ① $n>2$
✗ ② all vertices degree $> \frac{n}{2}$ (2.5 here)
No BUT still Hamiltonian

✗ ② all vertices degree $> \frac{n}{2}$ (2.5 here)
shows may not be a Hamiltonian circuit as we found.

Find all circuits and paths – Euler or Hamilton…



*Tip: Find degree of each vertex first!

- No Euler circuit because not all even degree vertices.
- No Euler path because not exactly 2 odd degree vertices
- Hamiltonian Circuit
  yes. Example: ABCDEA
- Hamiltonian Path
  yes. Example: ABCDE

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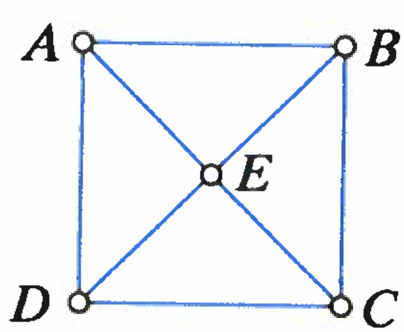
Find all circuits and paths – Euler or Hamilton… ANSWERS



No Euler Circuit or Path

Yes Hamilton Circuit: A,B,C,D,E,A

Yes Hamilton Path: A,B,C,D,E

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Drawing connections

- Note that if a graph has a **Hamilton circuit**, then it automatically has a **Hamilton path** – (the Hamilton circuit can always be truncated into a Hamilton path by dropping the last vertex of the circuit.)

- Contrast this with the mutually exclusive relationship between Euler circuits and paths: If a graph has an **Euler circuit** it **CANNOT** have an **Euler path** and vice versa.

Now try #1 in your HW! ← You Try

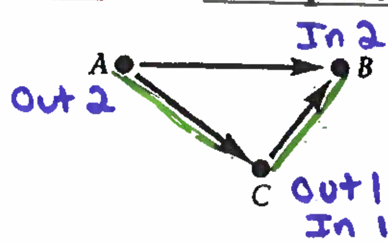
Packet p9 #1

Tournaments

A Tournament is a Complete Digraph.

every vertex adjacent to every other

edges with direction



* must be complete graph to do tournament so you know how head-to-head matchups work

*Remember: think of the arrow like a punch being thrown. Ex: AC is A beating C since arrow head "hits" C from A

One interesting characteristic of a complete digraph is that every tournament contains at least one Hamiltonian Path.

If there is exactly one Hamiltonian Path, it can be used to rank the teams in order from winner to loser.

Ex: Above, the Hamiltonian Path ACB

Shows A wins, C is 2nd place, B loses

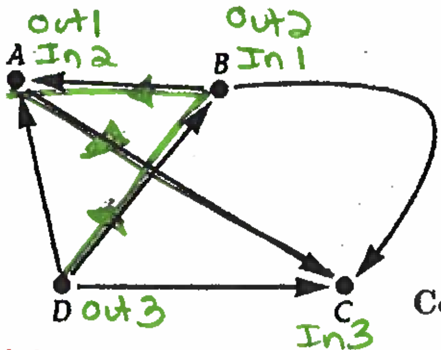
TIP: Do order with Highest outdegree to Lowest outdegree

Tournaments

Suppose four teams play in the school soccer round-robin tournament. The results of the competition follow:

Game	AB	AC	AD	BC	BD	CD
Winner	B	A	D	B	D	D

Draw a digraph to represent the tournament. Find a Hamiltonian path and use it to rank the participants from winner to loser.



This is a complete digraph.
because every vertex is adjacent to each other with an arrow

Is there only one Hamiltonian Path? Yes. DBAC

Rank the teams from first place to last. D, B, A, C → * Tip! Start with vertex with largest out degree, and go in order to lowest out degree vertex

Construct an adjacency matrix.

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 1 & 0 \\ B & 1 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 0 \\ D & 1 & 1 & 1 & 0 \end{array}$$

(Directed edge from B to A means that B beat A.)

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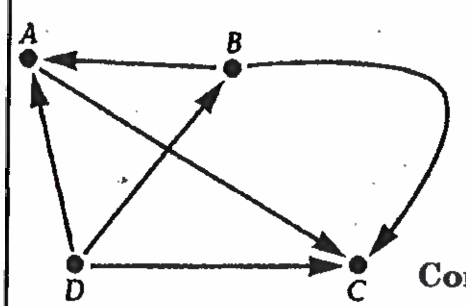
* In adjacency matrix for digraph, 0 shows loser and 1 shows winner.

Tournaments ANSWERS

Suppose four teams play in the school soccer round-robin tournament. The results of the competition follow:

Game	AB	AC	AD	BC	BD	CD
Winner	B	A	D	B	D	D

Draw a digraph to represent the tournament. Find a Hamiltonian path and use it to rank the participants from winner to loser.



This is a complete digraph.

Is there only one Hamiltonian Path? Yes!

Rank the teams from first place to last. D, B, A, C

Construct an adjacency matrix.

(Directed edge from B to A means that B beat A.)

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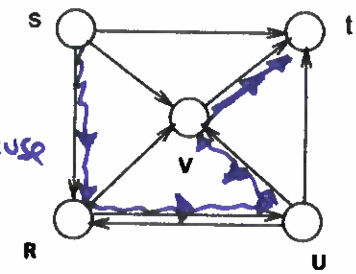
Can we rank RSTUV?

- **Does this have exactly one Hamilton path?** No. No ranking because
- **Is this complete?**

No. Every vertex is not adjacent to every other.
Ex: S and U aren't adjacent, R and t aren't adjacent



Hamiltonian Path: SRUVT
BUT not ranking Because not complete digraph

You Try

Now, Try HW # 6

Packet p7

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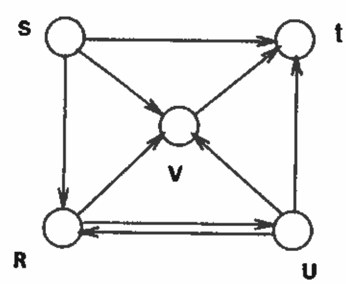
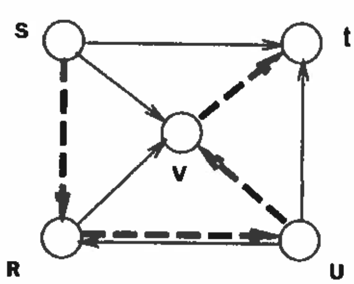
Can we rank RSTUV? ANSWERS

- **Does this have exactly one Hamilton path?**
- **Is this complete?**

- NO: No ranking
- Not complete because all vertices are not adjacent (S and U; R and T)





Now, Try HW # 6

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Packet p. 6-8

= Review & Practice of 4.1-4.5

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Homework

- Finish Review Packet p. 6-8
- Packet p. 9

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