Unit 6 Day 5

2.1-2.2 Fair Division – Discrete Case

Warm-Up

Consider a situation in which voters A, B, C, and D have 4, 3, 3, and 2 votes, respectively, and 7 votes are needed to pass an issue.

1. Identify all winning coalitions and their vote totals.
2. Find the power index for each voter.
3. Do the power indices reflect the distribution of votes?
4. Suppose the number of votes necessary to pass an issue is increased from 7 to 8. How does this change the power indices of the voters?

Warm-Up

Consider a situation in which voters A, B, C, and D have 4, 3, 3, and 2 votes, respectively, and 7 votes are needed to pass an issue.

1. Identify all winning coalitions and their vote totals.
   {A,B; 7}, {A,C; 7}, {A,B,C; 10}, {A,B,D; 9}
   {A,C,D; 9}, {B,C,D; 8}, {A,B,C,D; 12}
2. Find the power index for each voter.
   A: 5, B: 3, C: 3, D: 1
3. Do the power indices reflect the distribution of votes?
   No, A is higher and D's is lower.
4. Suppose the number of votes necessary to pass an issue is increased from 7 to 8. How does this change the power indices of the voters?
   {A,C,D; 9} {A,B,C; 10} {A,B,D; 9}
   {B,C,D; 8}, {A,B,C,D; 12}

new winning coalitions

Power Index
A: 3 C: 3
B: 3 D: 3
⇒ All voters have = power

Warm-Up ANSWERS

Consider a situation in which voters A, B, C, and D have 4, 3, 3, and 2 votes, respectively, and 7 votes are needed to pass an issue.

1. Identify all winning coalitions and their vote totals.
   A,B;7 | A,C;7 | A,B,C;10 | A,B,D;9 | A,C,D;9 | B,C,D;8 | A,B,C,D;12
2. Find the power index for each voter.
   A:5, B:3, C:3, D:1
3. Do the power indices reflect the distribution of votes?
   No, A's power is disproportionately high, while D's is low.
4. Suppose the number of votes necessary to pass an issue is increased from 7 to 8. How does this change the power indices of the voters?
   All voters now have equal power.

Homework Questions from last night?

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Tonight's Homework

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Notes
Unit 6 Day 6

2.1-2.2 Fair Division – Discrete Case

FAIR DIVISION

Consider these three scenarios.

1. Martha and Roy want to divide the last piece of the cake that their mother baked yesterday. Propose at least three methods of dividing the piece of cake that will seem fair to both Martha and Roy.

2. Juan and Mary are the only heirs to their mother's estate. The only object of significant value is the house in which they were raised. Propose at least three methods of resolving the issue of the disposition of the house that will seem fair to both Juan and Mary.

3. The sophomore, junior, and senior classes at Central High School have 333, 288, and 279 members, respectively. The school's student council is composed of 20 members divided among the three classes. Determine a fair number of seats on the council for each class.

Each group will consider their assigned fair-division scenario and present their solutions to the class.

FAIR DIVISION

There are many circumstances in which the division of an object or a set of assets in a fair way is important.

Examples of things that need to be fairly divided:

1. Food among children
2. A house in an estate among heirs
3. Seats in a governmental body among districts

In our next class, we will look at the Continuous case.

- Today, we'll look at the Discrete Case...

2.2: Estate Division




Fair Division can be DISCRETE or CONTINUOUS.

Discrete Division - occurs when the objects of the division cannot be meaningfully separated into pieces.

Ex) Dividing a house – a piece of a home is not useful!

Fair Division can be DISCRETE or CONTINUOUS.

Continuous Division - occurs when the objects of the division ***can*** be separated into pieces.

Ex) Dividing a cake
– infinite amount of ways!

- If the candies are identical and the children are entitled to equal shares, then the problem is simple. Each child gets two candies.

- If, on the other hand, the answer to either of the above questions is no, the solution is problematic.

- Should we, regardless of the type of candies involved and regardless of the children's relative entitlement, give each child two pieces of candy anyways?

- If we did, can we with assurance say we are dividing the candies "fairly"?
- What does "**fairly**" mean in this context.

What if.....

.....the candies are replaced by

- multi-million dollar paintings,
- jewelry,
- piece of land,
- seats in congress...

Fair Division can be DISCRETE or CONTINUOUS.

- It can be difficult to define fairness in some situations because different people place different values on the same object.

- Today, we'll look at the Discrete Case....

So ... What is a fair share?

- A **fair share** is any share that, in the opinion of the person getting it, is at least "one Nth" of the items to be divided where N is the number of players.
- The fair share is relative to the one receiving it. Anyone else's opinion doesn't matter. This implies only "proportional shares", not "envy free shares".
- Envy Free Shares: each person should feel that the received portion is at least as big as every other person's.

Estate Division

Estate Division is a DISCRETE case because the object cannot be meaningfully separated into pieces.

Dividing an estate is common in our society. It can be difficult and emotional for the parties involved.

In this section, we will learn the basic technique for dividing an estate.

Algorithm for Dividing an Estate:

1) Each heir submits a bid for each item in the estate. (Bids are not made on cash.)

2) A **fair share** is determined for each heir by finding the sum of his or her bids + cash & dividing this sum by the number of heirs.

3) Each item in the estate is given to the heir who bid the highest amount on that item.

4) Each heir is given an amount of cash from the estate that is equal to his or her fair share minus the amount the heir bid on the object(s) he or she received.

If this amount is negative, the heir pays that amount to the estate.

$fair share - $ from bids *won* = $ heir receives

5) The remaining cash in the estate is divided equally among the heirs.

Using this algorithm, each of the heirs receives a share that is larger than he or she thinks is fair.

Estate Division Example

Bo, Luke, and Daisy are 3 family members who are dividing the estate of their beloved Uncle Jesse.

Uncle Jesse's estate contains:

A Math book – Discrete Mathematics Through Applications
A DVD – The Notebook
$500 Cash
A car – 1969 Dodge Charger

Each family member will submit a "BID" for each *discrete* item. Not the Cash.

The bids show how much each person thinks/feels each item is worth to them personally.

Problem Continues Next ->

Estate Division Example

1. Initial Bids — $500 cash

(For now, we will round values to the nearest dollar)

	Book	DVD	Car	Fair Share = (sum of bids + cash)/heirs
Bo	$20	$30	$2,000	(20+30+2000+500)/3 = 850
Luke	$10	$10	$2,500	(10+10+2500+500)/3 = 1007
Daisy	$0	$5	$3,000	(0+5+3000+500)/3 = 1168

2. Determine each person's FAIR SHARE. Add their bids and the cash amount in the estate. Then, divide by the number of heirs sharing the estate.

3. Each item goes to the person who bid highest for that item.

Bo gets the Book and DVD
Luke gets NO items
Daisy gets the Car

Problem Continues Next ->

Estate Division Example

4. Each person will now receive cash to complete his fair share.

	Book	DVD	Car	Fair Share = (sum of bids + cash)/heirs
Bo	$20	$30	$2,000	(20+30+2000+500)/3 = $850
Luke	$10	$10	$2,500	(10+10+2500+500)/3 = $1007
Daisy	$0	$5	$3,000	(0+5+3000+500)/3 = $1168

Fair Share − $ of items received = $ owed

Bo is owed $850 − $50 = $800
Luke is owed $1007 − $0 = $1007
Daisy is owed $1168 − $3000 = −$1832 Daisy pays into the estate $1832

So, now the total cash in the estate is $500 + $1832 = $2332

Problem Continues Next ->

Estate Division Example

5. Pay off Bo and Luke with the cash in the estate.

Give Bo the $800 he is owed. The estate now has $2332 − 800 = $1532

Give Luke the $1007 he is owed. The estate now has $1532 − 1007 = $ 525

6. The remaining cash, $525, is now divided equally among the 3 heirs.

$525 / 3 = $175 to each.

Problem Continues Next ->

Estate Division Example

7. So, now finalize who got what.

Bo got Math Book, DVD, $800 + $175 = $975
(The rest of Bo's fair share; Bo's share of the Cash left in the estate)

Luke got $1007 + $175 = $1182

Daisy got Car −$1832 + $175 = $ −$1657
(So, Daisy got the car, but paid into the estate $1657)

Notice that each heir got more than their fair share. In the end each heir should feel that they were treated fairly.

Another Example

Monica, Ross, and Chandler are heirs to an estate that includes a house, a boat, a car and $150,000 cash.

Here are their initial bids:

	House	Boat	Car	Fair Share
Monica	$80,000	$5,000	$8,000	(80K+5K+8K+150K)/3 = 81K
Ross	$70,000	$9,000	$11,000	(70K+9K+11K+150K)/3 = 80K
Chandler	$76,000	$7,000	$13,000	(76K+7K+13K+150K)/3 = 82K

Find the fair share for each heir. Round amounts to the nearest dollar.

Problem Continues Next ->

Another Example

	House	Boat	Car	Fair Share
Monica	$80,000	$5,000	$8,000	(80,000+5,000+8,000+150000)/3 = $81,000
Ross	$70,000	$9,000	$11,000	(70,000+9,000+11,000+150000)/3 = $80,000
Chandler	$76,000	$7,000	$13,000	(76,000+7,000+13,000+150000)/3 = $82,000

Who got what item? Whoever bid highest for it

Monica: House
Ross: Boat
Chandler: Car

Who owes money? Who is owed money?

Monica : 81000 − 80000 (House Bid) = 1,000
Ross : 80000 − 9000 (Boat Bid) = 71,000
Chandler : 82000 − 13000 (car bid) = 69,000

Problem Continues Next ->

$ to complete their fair share

Another Example ANSWERS

	House	Boat	Car	Fair Share
Monica	$80,000	$5,000	$8,000	(80,000+5,000+8,000+150000)/3 = $81,000
Ross	$70,000	$9,000	$11,000	(70,000+9,000+11,000+150000)/3 = $80,000
Chandler	$76,000	$7,000	$13,000	(76,000+7,000+13,000+150000)/3 = $82,000

Who got what item?

Monica – House
Ross – Boat
Chandler – Car

Who owes money? Who is owed money?

Monica : $81,000 − $80,000 = $1,000
Ross : $80,000 − $9,000 = $71,000
Chandler : $82,000 − $13,000 = $69,000

Problem Continues Next ->

Another Example

The estate has $150,000 cash.

~~Monica : $81,000 – $80,000 = $1,000~~
Ross : $80,000 – $9,000 = $71,000
Chandler : $82,000 – $13,000 = $69,000

Now, pay off everyone who is owed money.

Give Monica the $1,000 she is owed. The estate now has $150,000 − 1,000 = $149,000

Give Ross the $71,000 he is owed. The estate now has $149,000 − 71,000 = $78,000

Give Chandler the $69,000 he is owed. The estate now has $78,000 − 69,000 = $9,000

The remaining $9,000 is divided equally among the 3 heirs.

$9,000 / 3 = $3,000 to each.

Problem Continues Next ->

Called "Remaining" Cash or "Bonus" cash

Another Example

Finalize who got what.

Monica got	House, $1,000 + $3,000	= $4,000
Ross got	Boat $71,000 + $3,000	= $74,000
Chandler got	Car $69,000 + $3,000	= $72,000

often called "Final Settlement"

You Try

Zack, Slater, Jessie, and Lisa are heirs to the estate of their favorite principal, Mr. Belding. Mr. Belding leaves behind a closet full of men's clothes, a Yacht, a Book of Secrets and $10,000 cash. The bids of the heirs are detailed in the matrix below. Show who gets what item and how much cash each will receive/pay. Round money amounts to the nearest cent.

	Clothes	Yacht	Book of Secrets
Zack	$100	$14,000	$5,000
Slater	$150	$12,000	$2,500
Jessie	$100	$16,000	$4,000
Lisa	$500	$10,000	$2,000

Fair Share
(100 + 14000 + 5000 + 10000)/4 = 7275.00
(150 + 12000 + 2500 + 10000)/4 = 6162.50
(100 + 16000 + 4000 + 10000)/4 = 7525.00
(500 + 10000 + 2000 + 10000)/4 = 5625.00

Final Settlement

Zack	item(s) Book of secrets	cash	3503.13
Slater	item(s)	cash	7390.63
Jessie	item(s) Yacht	cash	−7246.87
Lisa	item(s) Clothes	cash	6353.13

2275 + 1228.13
6162.50 + 1228.13
−8475 + 1228.13
5125 + 1228.13

Owed/owes to Estate

Zack: 7275 − 5000 (Book of secrets) = 2275 owed by estate
Slater: 6162.50 − 0 stuff = 6162.50 owed by estate
Jessie: 7525 − 16000 yacht = −8475 owes to estate (owes to because ⊖ due to getting a pricey inheritance item)
Lisa: 5625 − 500 clothes = 5125 owed by estate

$ in/out of estate

10000 cash at start + 8475 from Jessie = 18475 in estate now

18475 in estate now − 2275 to Zach − 6162.50 to Slater − 5125 to Lisa = 4912.50 now in estate

Remaining Cash 4912.50/4 = 1228.13 to each heir

UNEVEN ESTATE DIVISION

Step 1 - Bids

- There are, at times, division of an estate where not everyone shares equally in the distribution. In these instances, we change part of our algorithm to reflect the differences.
- Just like before, each heir submits a bid for each item in the estate. (Bids are NOT made on cash in the estate because it can be divided equally without controversy.)

Step 2 – Fair Share

- A fair share is determined for each heir. In this case, find the sum of his/her bids and any cash and **multiply by the fractional proportion** (or percentage).

This will ensure that each heir receives the correct proportion.

Step 3 – Highest Bidder Wins

- Each item in the estate is given to the heir who bid the highest on that item.

Step 4 – Cash Division

- Each heir is given an amount of cash that is equal to his/her fair share (from step 2) less the amount the heir bid on the object(s) he/she received. If this amount is negative, the heir pays that amount into the estate.

Step 5 – Remaining $$

- The remaining cash is divided among the heirs. This is achieved by multiplying the remaining cash by the fractional proportion.

Example: Wendy, Keith, & Rhonda are heirs to an estate that includes a house, a tractor, a diamond ring, and a car. It also includes $300,000 in cash. Wendy and Keith both receive ¼ of the estate, while Rhonda receives ½ of the estate.

- Step 1: Each heir submits a bid for the house, tractor, ring, and car.

	House	Tractor	Diamond Ring	Car
Wendy	$120,000	$3,000	$5,000	$8,000
Keith	$115,000	$5,000	$2,000	$9,000
Rhonda	$125,000	$4,000	$3,000	$8,500

Problem Continues Next ->

Wendy and Keith both receive ¼ of the estate. Rhonda receives ½ of the estate. The estate includes $300,000 in cash.

	House	Tractor	Diamond Ring	Car
Wendy	$120,000	$3,000	$5,000	$8,000
Keith	$115,000	$5,000	$2,000	$9,000
Rhonda	$125,000	$4,000	$3,000	$8,500

Step 2: A fair share is determined for each heir

Wendy: ($120,000 + $3,000 + $5,000 + $8,000 + $300,000) x ¼ = $109,000
Keith: ($115,000 + $5,000 + $2,000 + $9,000 + $300,000) x ¼ = $107,750
Rhonda: ($125,000 + $4,000 + $3,000 + $8,500 + $300,000) x ½ = $220,250

Problem Continues Next ->

- Step 3:

Wendy receives the diamond ring
Keith receives the tractor and the car
Rhonda receives the house

	House	Tractor	Diamond Ring	Car
Wendy	$120,000	$3,000	$5,000	$8,000
Keith	$115,000	$5,000	$2,000	$9,000
Rhonda	$125,000	$4,000	$3,000	$8,500

Problem Continues Next ->

	House	Tractor	Diamond Ring	Car
Wendy	$120,000	$3,000	$5,000	$8,000
Keith	$115,000	$5,000	$2,000	$9,000
Rhonda	$125,000	$4,000	$3,000	$8,500

Step 4: Cash equal to the difference between the fair share and the value of the awarded items is given to each heir.

Fair share − what got = $ owed to/by estate

Wendy: $109,000 - $5,000 = $104,000
Keith: $107,750 - $14,000 = $93,750
Rhonda: $220,250 - $125,000 = $95,250

Problem Continues Next ->

$ in/out of estate
300000 − 104000 − 93750 − 95250 = 7000
mestate Wendy Keith Rhonda remaining cash

Step 5:

The cash given to the heirs totals $293,000 which leaves $300,000 - $293,000 = $7,000 cash in the estate. Each heir receives a bonus of :

Wendy $7,000 x ¼ = $1,750
Keith $7,000 x ¼ = $1,750
Rhonda $7,000 x ½ = $3,500

Use fractional proportion

Problem Continues Next ->

Finalize who got what.

Cash Bonus (remaining cash)

Wendy received Ring, $104,000 + $1,750 = $105,750
Keith received Tractor, Car, $93,750 + $1,750 = $95,500
Rhonda received House, $95,250 + $3,500 = 98,750

Classwork/Homework

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