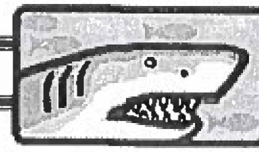
Key

UNIT 2 REVIEW – Leslie Model, Markov Chains, Game Theory



1. Suppose a shark population has the characteristics described in the table below.

AGE (years)	0-5	5-10	10-15	15-20	20-25	25-30	30-35
BIRTRATE	0	0	1.2	0.8	0.7	0.2	0
SURVIVAL RATE	0.5	0.8	0.9	0.9	0.7	0.5	0
INITIAL POPULATION	10	14	25	30	20	15	8

skip last 0 Surviv rate in Leslie matrix

a. What is the maximum age of this animal? 35 years

b. Construct the LESLIE MATRIX for this animal.

$$L = \begin{bmatrix} 0 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .8 & 0 & 0 & 0 & 0 \\ 1.2 & 0 & 0 & .9 & 0 & 0 & 0 \\ .8 & 0 & 0 & 0 & .9 & 0 & 0 \\ .7 & 0 & 0 & 0 & 0 & .7 & 0 \\ .2 & 0 & 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7 x 7

Birthrates in 1st column

Survival Rates go into Super Diagonal EXCEPT the last 0 one.

c. Find the population distribution after 150 years (30 cycles). Show work!

$P_0 L^{30} = [59.4 \quad 29.4 \quad 23.3 \quad 20.7 \quad 12.8 \quad 6.3]$

d. Find the TOTAL population after 150 years. Show work!

→ 30 cycles

$P_0 L^{30} \cdot C = \boxed{170.5}$

e. Determine the LONG TERM GROWTH RATE for this population. Show work!

$(P_0 \cdot L^{50})C = P_{50} = 209.34$

$(P_0 \cdot L^{51})C = P_{51} = 211.5$

$\frac{211.5 - 209.34}{209.34} = .0103$

$P_{52} = 213.68$

$\frac{213.68 - 211.5}{211.5} = .0103$

$\boxed{1.03\%}$

f. If the maximum sustainable population for this population in its native habitat is 500, when will the maximum population be reached (answer in cycles and years)?

During cycle 134 or $\boxed{670 \text{ years}}$

2. Given that a person's last soda purchase was Coke, there is a 90% chance that his next soda purchase will also be Coke and an equal chance between Pepsi and Sprite. If a person's last soda purchase was Pepsi, there is an 80% chance that his next soda purchase will also be Pepsi and he will always choose Coke over Sprite. If a person's last soda purchase was Sprite, there is a 70% chance that his next purchase will also be Sprite, 20% chance it will be Coke and the remaining choose Pepsi.

a. Create a transition matrix for this situation and label the rows and columns.

$$T = \begin{array}{c} \\ C \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & P & S \end{array} \\ \begin{bmatrix} .9 & .05 & .05 \\ .2 & .8 & 0 \\ .2 & .1 & .7 \end{bmatrix} \end{array}$$

b. Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi three purchases from now? Show the matrix equation.

$D_0 = [1 \quad 0 \quad 0]$ (C, P, S)

$D_3 = D_0 T^3 = [.781 \quad .1215 \quad .0975]$ (C, P, S)

$\boxed{12.15\%}$

c. In the long run, what is the probability that a person will purchase Sprite if they are currently a Pepsi purchaser?

$D_0 = [0 \quad 1 \quad 0]$ (C, P, S)

$D_0 T^{20} = D_{20} = [.667 \quad .22 \quad .11]$

$D_0 T^{30} = D_{30} = [.667 \quad .22 \quad .11]$

$\boxed{11\%}$

3. A certain species of insect lives 5 weeks. Through each of the first 4 weeks only 60% survive into the next week. Each female bug that survives into the 5th week produces, on average, 20 new female bugs. Bugs that only survive the first four weeks do not produce offspring.

a. Construct a Leslie matrix representing the life cycle of these bugs.

$$P_0 = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ [\,0 & 0 & 0 & 0 & 10\,] \end{array} \qquad L = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & .6 & 0 & 0 & 0 \\ 2 & 0 & 0 & .6 & 0 & 0 \\ 3 & 0 & 0 & 0 & .6 & 0 \\ 4 & 0 & 0 & 0 & 0 & .6 \\ 5 & 20 & 0 & 0 & 0 & 0 \end{array}$$

10 female bugs in their 5th week of life move into your garage.

b. What is the newborn population after the 6th cycle? $P_0 L^6 = \boxed{518.4}$ ← 1st # in matrix

c. What will the total population be after 11 cycles? $P_0 L^{11} \cdot C = \boxed{1343.7}$ where $C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Suppose the initial population looked like this $P_0 = [\,0 \quad 20 \quad 15 \quad 25 \quad 5\,]$,

d. What would the total population be after 22 cycles? $P_0 \cdot L^{22} \cdot C = 16818.33$

e. What is the growth rate between P_{21} and P_{22}?

$P_{21} = 6138.7 \qquad P_{22} = 16818.33$

$GR = \dfrac{\text{new} - \text{prev}}{\text{prev}} = \dfrac{16818.33 - 6138.7}{6138.7} = 1.74 \qquad \boxed{174\%}$

4. Each of the following matrices represents a payoff matrix for a game. Determine the best strategy for the row and column players. If the game is strictly determined, find the saddle point of the game. If the game is not strictly determined, explain why not.

a)

$$\begin{array}{c|cc|c} & C & D & \text{maximin} \\ A & -2 & 5 & -2 \\ B & 2 & 3 & 2 \\ \hline \text{minimax} & 2 & 5 & \end{array}$$

Row strategy: B

Column strategy: C

Strictly Determined?: yes/no (yes)

Saddle Point (if yes above): 2

b)

$$\begin{array}{c|ccc|c} & E & F & G & \text{maximin} \\ A & 2 & -1 & 2 & -1 \\ B & -3 & -4 & -1 & -4 \\ C & 4 & 1 & 2 & 1 \\ D & 3 & 1 & -2 & -2 \\ \hline \text{minimax} & 4 & 1 & -2 & \end{array}$$

1st Eliminate row B because all values < row C

2nd Eliminate column E because remaining values > column F

Row strategy: C

Column strategy: F

Strictly Determined?: yes/no (yes)

Saddle Point(if yes above): 1

5. Suppose that Gary and Liz play the "penny matching game" and use the following payoff matrix.

$$\begin{array}{cc|cc|c} & & \text{LIZ} & & \\ & & H & T & \\ \text{GARY} & H & 3 & -2 & -2 \\ & T & -2 & 1 & -2 \\ & & 3 & 1 & \end{array} \quad \text{maximin} \quad \text{minimax}$$

$[p \;\; 1-p]\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$: $3p - 2 + 2p = -2p + 1 - p$; $5p - 2 = -3p + 1$; $8p = 3$; $p = 3/8$; $1 - p = 5/8$

$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} q \\ 1-q \end{bmatrix}$: $3q - 2 + 2q = -2q + 1 - q$; $5q - 2 = -3q + 1$; $8q = 3$; $q = 3/8$; $1 - q = 5/8$

It's just a coincidence that p=q here

a. Use the row matrix $[p \quad 1-p]$ to find Gary's best strategy for this game. $[\,3/8 \quad 5/8\,]$ Gary should do heads 3/8 of the time and tails 5/8.

b. Use the column matrix $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ to find Liz's best strategy for this game. $\begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$ Liz should also do

c. What is Gary's expectation for this game, provided that both players follow their "best strategy"? Interpret your answer in terms of how many pennies Gary can expect to win/lose over a number of games?

$$[\,3/8 \quad 5/8\,]\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} = \boxed{-.125}$$

Gary will lose 12.5 cents every 100 times

(or Gary will lose about 0.125 pennies per game)

6. A supplier provides coffee beans to shops A, B, and C in 1 pound, 2 pound, and 3 pound bags, as shown in the table below.

	1 pound	2 pound	3 pound
Shop A	50	100	30
Shop B	60	150	40
Shop C	80	200	70

The cost of a 1 pound, 2 pound, and 3 pound bag are $10.60, $17.20, and $22.50 respectively.

a) Use matrices to calculate the total cost of the coffee bean purchases at each shop.

$$\begin{array}{cc} & \begin{array}{ccc} 1\text{ lb} & 2\text{ lb} & 3\text{ lb} \end{array} \\ \begin{array}{c} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{array} & \begin{bmatrix} 50 & 100 & 30 \\ 60 & 150 & 40 \\ 80 & 200 & 70 \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \text{cost (\$)} \\ \begin{array}{c} 1\text{ lb} \\ 2\text{ lb} \\ 3\text{ lb} \end{array} & \begin{bmatrix} 10.60 \\ 17.20 \\ 22.50 \end{bmatrix} \end{array} = \boxed{\begin{array}{cc} & \text{Total Cost (\$)} \\ \begin{array}{c} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{array} & \begin{bmatrix} \$29{,}25 \\ \$4{,}116 \\ \$5{,}863 \end{bmatrix} \end{array}}$$

$$\underset{\text{Shops lbs}}{(3 \times 3)} \cdot \underset{\text{lbs cost}}{(3 \times 1)}$$

b) Suppose the supplier provides a 15% discount. Use matrices to determine what the costs of the bean purchases will be.

$$\begin{array}{cc} & \text{cost} \\ \begin{array}{c} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{array} & \begin{bmatrix} 2925 \\ 4116 \\ 5863 \end{bmatrix} \end{array} \cdot 0.85 = \boxed{\begin{array}{cc} & \text{Total Bean Costs (\$)} \\ \begin{array}{c} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{array} & \begin{bmatrix} 2{,}486.25 \\ 3{,}498.60 \\ 4{,}983.55 \end{bmatrix} \end{array}}$$

at Discount of 15%

1 − .15 because 15% discount means pay 85% :)

7. A quilt maker plans to make 14 quilts this year from 113 yards of fabric. A small quilt requires 4 yards of fabric. A medium quilt requires 7 yards of fabric, and a large quilt requires 11 yards of fabric. She plans to make twice as many large quilts as small quilts. How many of each type of quilt should she make?

Define your variables:

x = # of small quilts
m = # of medium quilts
l = # of large quilts

System of linear equations:

$x + m + l = 14$ (14 quilts this year)
$4x + 7m + 11l = 113$ (yards of fabric)
$l = 2x$ (twice as many large as small)
$-2x + 0m + 1l = 0$

Matrices work:

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 7 & 11 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ m \\ l \end{bmatrix} = \underset{B}{\begin{bmatrix} 14 \\ 113 \\ 0 \end{bmatrix}}$$

then do $A^{-1} \cdot B$ in calc.

Solution:

The quilt maker should make 3 small, 5 medium, and 6 large quilts this year.

8. The cost of admission to a popular music concert was $162 for 12 children and 3 adults. The admission was $122 for 8 children and 3 adults at another music concert. How much was the admission for each child and adult?

Define Variables:

c = admission cost for a child
a = admission cost for an adult

Linear Equations:

$12c + 3a = 162$
$8c + 3a = 122$

Matrices work:

$$\underset{A}{\begin{bmatrix} 12 & 3 \\ 8 & 3 \end{bmatrix}} \begin{bmatrix} c \\ a \end{bmatrix} = \underset{B}{\begin{bmatrix} 162 \\ 122 \end{bmatrix}}$$

then do $A^{-1} \cdot B$ in calculator

The admission cost is $10 per child and $14 per adult.

Space = 0 A=1 B=2 C=3 D=4 E=5 F=6 G=7 H=8 I=9 J=10 K=11 L=12 M=13
N=14 O=15 P=16 Q=17 R=18 S=19 T=20 U=21 V=22 W=23 X=24 Y=25 Z=26

9. Encode the message NEW YORK YANKEES using the encoding matrix $\begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 4 & -3 & 2 \end{bmatrix}$. Show work.

14 - 5 - 23 - 0 - 25 - 15 - 18 - 11 - 0 - 25 - 1 - 14 - 11 - 5 - 5 - 19
N - E - W - _ - Y - O - R - K - _ - Y - A - N - K - E - E - S

$$\underbrace{\begin{bmatrix} 14 & 5 & 23 \\ 0 & 25 & 15 \\ 18 & 11 & 0 \\ 25 & 1 & 14 \\ 11 & 5 & 5 \\ 19 & 0 & 0 \end{bmatrix}}_{\text{Extra zeros finish off the matrix after the message}} \cdot \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 4 & -3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 124 & -83 & 51 \\ 10 & -45 & 55 \\ 32 & -18 & 11 \\ 129 & -67 & 29 \\ 43 & -26 & 15 \\ 57 & -19 & 0 \end{bmatrix}}$$

10. Decode the message -2 -24 -19 17 -15 17 0 -36 -5 7 0 -38 -15 -3 using the encoding matrix $\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$. Show work.

you need inverse to decode !! ☺

$$\begin{bmatrix} -2 & -24 \\ -19 & 17 \\ -15 & 17 \\ 0 & -36 \\ -5 & 7 \\ 0 & -38 \\ -15 & -3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 15 \\ 19 & 20 \\ 15 & 14 \\ 0 & 18 \\ 5 & 4 \\ 0 & 19 \\ 15 & 24 \end{bmatrix}$$

THEN you must finish by getting the message!

2 - 15 - 19 - 20 - 15 - 14 - 0
B O S T O N _

18 - 5 - 4 - 0 - 19 - 15 - 24
R E D _ S O X

$\boxed{\text{BOSTON RED SOX}}$