GAME THEORY
Day 8

Section 7.4

GOOD MORNING

Grab <u>one</u> penny.

I will walk around and check your HW.

Warm Up

A school categorizes its students as distinguished, accomplished, proficient, and developing. Data show that the school's students move from one category to another according to the probabilities shown in the transition matrix:

$$T = \begin{array}{c} \\ Dis \\ Ac \\ \text{Pr} \\ Dev \end{array} \begin{array}{c} \begin{array}{cccc} Dis & Ac & Pr & Dev \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0.2 & 0.4 & 0.4 \end{bmatrix} \end{array}$$

1) Write an initial-state matrix for a student who enters the school as proficient.
2) If students are reclassified weekly, predict the student from part a's future after one month in the school.
3) What can we expect long-term for each type of student at this school?

Warm Up ANSWERS

A school categorizes its students as distinguished, accomplished, proficient, and developing. Data show that the school's students move from one category to another according to the probabilities shown in the transition matrix:

$$T = \begin{array}{c} \\ Dis \\ Ac \\ \mathrm{Pr} \\ Dev \end{array} \begin{array}{c} \begin{array}{cccc} Dis & Ac & Pr & Dev \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0.2 & 0.4 & 0.4 \end{bmatrix} \end{array}$$

1) Write an initial-state matrix for a student who enters the school as proficient.

$D_o = [\ 0 \quad 0 \quad 1 \quad 0\ ]$

2) If students are reclassified weekly, predict the student from part a's future after one month in the school.

$D_4 = D_oT^4 = [\ 0.629 \quad 0.201 \quad 0.131 \quad 0.039\ ]$;

so a 62.9% chance of being distinguished!

3) What can we expect long-term for each type of student at this school? $T^{20} = T^{30} = T^{40}$ where the first column is around 1.

All students eventually will be distinguished!

Any Homework Questions?

Last night's HW was
Finish Packet p. 6 -7
AND Packet p. 9

Today's Classwork and Homework

Classwork

- Packet p.12 #1 & 2

Homework

- Packet p. 10
- And Packet p. 8

Notes Day 8
GAME THEORY Strictly Determined

Section 7.4

GAME THEORY

We think of games as fun and relaxing ways to spend our time.

However, there are many <u>DECISION-MAKING</u> situations in fields such as economics and politics that can also be thought of as games.

<u>GAME THEORY</u> is a <u>DECISION-MAKING</u> technique.

Players in these games may be individuals, teams of people, companies, markets, even whole countries who have <u>CONFLICTING INTERESTS</u>.

In these situations, each player has a set of alternative courses of action called <u>STRATEGIES</u> that can be used in making decisions.

Mathematical Game Theory deals with selecting the best strategies for a player to follow in order to achieve his most favorable outcome.

What do you think these words mean?

- **Maximin**
- It is the maximum of all the minimums!

- **Minimax**
- It is the minimum of all the maximums!

GAME THEORY

Coin Game – Two Players (Player-R & Player-C)

→ Row player (R) and Column player (C)

Both players simultaneously display a coin.

- This is not a random flip. The player chooses which side to display.

If both players display heads, then Player-R wins 3¢ from Player-C.

If both players display tails, then Player-R pays Player-C 2¢.

If one player displays a head and the other displays a tail, then Player-R pays Player-C 1¢.

What is the best STRATEGY for each player ?

Let's play for 3 minutes.

Wait....could matrices help us out here?

GAME THEORY

If both players display heads, then Player-R wins 3¢ from Player-C.
If both players display tails, then Player-R pays Player-C 2¢.

If one player displays a head and the other displays a tail, then Player-R pays Player-C 1¢.

Our text describes Sol and Tina playing our coin game. Sol is Player-R and Tina is Player-C.

If you figured out the game, you should have found that this game isn't such a good deal for Sol (the row player). As long as Tina plays tails she cannot lose.

If Sol knows that Tina is going to play tails, he should display heads because he will lose more if he doesn't. He should minimize his losses.

This is a rather boring game because both players will do the same thing every time.

A game in which the best strategy for both players is to pursue the same strategy every time is called STRICTLY DETERMINED.

GAME THEORY

If both players display heads, then Player-R wins 3¢ from Player-C.
If both players display tails, then Player-R pays Player-C 2¢.

If one player displays a head and the other displays a tail, then Player-R pays Player-C 1¢.

Although **strictly determined** games are fairly boring, there are situations in life in which they cannot be avoided and knowing how to analyze them properly can be beneficial. Strictly determined games are often very simple, but they can be difficult to analyze without an organizational scheme. **Matrices** offer a way of doing this.

The following matrix represents Sol's view of the game.

***It is customary to write a game matrix from the viewpoint of the player associated with the matrix rows. Such a matrix is called a PAYOFF MATRIX.**

$$\begin{array}{ccc} & & \text{Tina} \\ & & \begin{array}{cc}\text{Heads} & \text{Tails}\end{array} \\ \text{Sol} & \begin{array}{c}\text{Heads} \\ \text{Tails}\end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \end{array}$$

The entries are the payoffs to Sol for each outcome of the game.

GAME THEORY

Consider the game from Sol's point of view. He wants to minimize his loses.

If he plays heads, the worst he can do is lose 1 cent. If he plays tails the worst he can do is lose 2 cents.

It's better to lose 1 cent than 2 cents, so Sol should play heads.

$$\begin{array}{ccc} & & \text{Tina} & & \\ & & \text{Heads} & \text{Tails} & \text{Row Minimums} \\ \text{Sol} & \begin{array}{c}\text{Heads}\\ \text{Tails}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}} & \begin{array}{c}-1\\ -2\end{array} \\ \text{Column Maximums} & & 3 & -1 & \end{array}$$

In general, the best strategy for the <u>row player</u> in a strictly determined game is to select the largest of the row minimums.

This is called the <u>MAXIMIN</u>. (maximum of the row minimums)

Consider the game from Tina's point of view. She wants to do the opposite of Sol since she's the column player. She must view minimums as maximums and vice-versa.

In general, the best strategy for the <u>column player</u> in a strictly determined game is to select the smallest of the column maximums.

This is called the <u>MINIMAX</u>. (minimum of the column maximums)

GAME THEORY

$$
\begin{array}{cc|cc|c}
 & & \multicolumn{2}{c|}{\text{Tina}} & \text{Row} \\
 & & \text{Heads} & \text{Tails} & \text{Minimum} \\
\text{Sol} & \text{Heads} & 3 & -1 & -1 \\
 & \text{Tails} & -1 & -2 & -2 \\
\text{Column Maximum} & & 3 & -1 &
\end{array}
$$

$$
\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}
$$

In this game the value selected by both players is the –1 in the upper right corner.

A STRICTLY DETERMINED game is one in which the MAXIMIN and MINIMAX are the same.

That value is called the <u>SADDLE POINT</u>.

Practice

1. Each of the following matrices represents a payoff matrix for a game. Determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game.
   If the game is not strictly determined, explain why.

$$a.\ \begin{pmatrix} 16 & 8 \\ 12 & 4 \end{pmatrix}$$

$$b.\ \begin{pmatrix} 0 & 4 \\ -1 & 2 \end{pmatrix}$$

$$c.\ \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$$

Practice Answers

1. For each payoff matrix, determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game. If the game is not strictly determined, explain why.

$$a.\ \begin{array}{cc} \begin{pmatrix} 16 & 8 \\ 12 & 4 \end{pmatrix} & \begin{array}{l} 8 \text{ maximin} \\ 4 \end{array} \\ \begin{array}{cc} 16 & 8 \text{ minimax} \end{array} & \end{array}$$

*Best strategies are Row 1 and Column 2.

*Saddle Point at 8.

$$b.\ \begin{array}{cc} \begin{pmatrix} 0 & 4 \\ -1 & 2 \end{pmatrix} & \begin{array}{l} 0 \text{ maximin} \\ -1 \end{array} \\ \begin{array}{cc} \text{minimax } 0 & 4 \end{array} & \end{array}$$

*Best strategies are Row 1 and Column 1.

*Saddle Point at 0.

$$c.\ \begin{array}{cc} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} & \begin{array}{l} -3 \\ -3 \end{array} \\ \begin{array}{cc} \text{minimax } 2 & 4 \end{array} & \end{array}$$

← doesn't matter which they choose

*Not a Strictly Determined Game because maximin does not = minimax

Practice

2. a. For a game defined by the following matrix, determine the best strategies for the row and column players and the saddle point of the game.

$$\begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$$

b. Add 4 to each element in the matrix given in part a. How does this affect the best strategies and the saddle point of the game?

c. Multiply each element in the matrix given in part a by 2. How does this affect the best strategies and saddle point of the game?

d. Make a conjecture.

Practice Answers

2. a. For a game defined by the following matrix, determine the best strategies for the row and column players and the saddle point of the game.

$$\begin{array}{cl} \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix} & \begin{array}{l} -4 \\ 3 \ \text{maximin} \end{array} \\ \begin{array}{cc} 5 & \quad 3 \ \text{minimax} \end{array} & \end{array}$$

*Saddle Point at 3
*Best strategies are Row 2 and Column 2.

b. Add 4 to each element in the matrix given in part a. How does this affect the best strategies and the saddle point of the game? *Saddle Point at 7. *No effect on best strategies.

c. Multiply each element in the matrix given in part a by 2. How does this affect the best strategies and saddle point of the game? *Saddle Point at 6. *No effect on best strategies.

d. Make a conjecture.

*Adding a set value or multiplying a set value to each element of a payoff matrix changes the saddle point in that same manner BUT has no effect on the best strategies.

Practice

3. The Democrats and Republicans are engaged in a political campaign for mayor in a small midwestern community. Both parties are planning their strategies for winning votes for the candidate in the final days. The Democrats have settled on two strategies, A and B, and Republicans plan to counter with strategies C and D. A local newspaper got wind of their plans and conducted a survey of eligible voters.

The results of the survey show that…

If the Democrats choose plan A and the Republicans choose C, then the Democrats will gain 150 votes.

If the Democrats choose plan A and the Republicans choose D, the Democrats will lose 50 votes.

If the Democrats choose B and the Republicans choose C, the Democrats will gain 200 votes.

If the Democrats choose B and the Republicans choose D, the Democrats will lose 75 votes.

Write this information as a matrix game. Find the best strategies and the saddle point of the game.

Practice Answers

3. The Democrats and Republicans are engaged in a political campaign for mayor in a small midwestern community. Both parties are planning their strategies for winning votes for the candidate in the final days. The Democrats have settled on two strategies, A and B, and Republicans plan to counter with strategies C and D. A local newspaper got wind of their plans and conducted a survey of eligible voters. The results of the survey show that…

if Dems choose A and Repubs choose C, then Dems will gain 150 votes.

If Dems choose A and Repubs choose D, the Dems will lose 50 votes.

If Dems choose B and Repubs choose C, the Dems will gain 200 votes.

If Dems choose B and Repubs choose D, the Dems will lose 75 votes.

Write this information as a matrix game. Find the best strategies and the saddle point of the game.

$$\begin{array}{ccc} & & \textit{Repubs} \\ & & \begin{array}{cc} C & D \end{array} \\ \textit{Dems} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 150 & -50 \\ 200 & -75 \end{pmatrix} \end{array} \begin{array}{l} \mathbf{-50} \text{ maximin} \\ \mathbf{-75} \end{array}$$

$$\begin{array}{ccc} \mathbf{200} & \mathbf{-50} & \text{minimax} \end{array}$$

*Saddle Point at -50 votes.

*Best strategies are for the Democrats to choose plan A and Republicans plan D.

GAME THEORY

When players have more than two strategies, a game is harder to analyze. It is helpful to eliminate strategies that are DOMINATED by other strategies. A strategy DOMINATES another if it is always a better choice.

In a competition between Dino's Pizza and Sal's Pizza, both are considering four strategies:

1) running no special,
2) offering a free minipizza with the purchase of a large pizza,
3) offering a free medium pizza with the purchase of a large one, and
4) offering a free drink with any pizza purchase.

A market study estimates the gain in dollars per week to Dino's over Sal's according to the following matrix. What should the two restaurants do?

		Sal's			
		No special	Mini	Medium	Drink
Dino's	No special	200	−400	−300	−600
	Mini	500	100	200	600
	Medium	400	−100	−200	−300
	Drink	300	0	400	−200

See next slide for how to do…

GAME THEORY

$$
\begin{array}{ll}
 & \text{Sal's} \\
\text{Dino's} &
\begin{array}{l} \\ \text{No special} \\ \text{Mini} \\ \text{Medium} \\ \text{Drink} \end{array}
\begin{array}{c}
\begin{array}{cccc} \text{No special} & \text{Mini} & \text{Medium} & \text{Drink} \end{array} \\
\begin{bmatrix} 200 & -400 & -300 & -600 \\ 500 & 100 & 200 & 600 \\ 400 & -100 & -200 & -300 \\ 300 & 0 & 400 & -200 \end{bmatrix}
\end{array}
\end{array}
$$

Notice, that from Dino's point of view, no matter what Sal decides to do, Dino always gets a <u>higher</u> payoff from row 2 (minipizza) than from row 1 (no special) because each value in row 2 is larger than the corresponding one in row 1.

→ It would make no sense for Sal to run No Special. Row 2 DOMINIATES row 1, so draw a line through row 1.

Are any other rows dominated?

Row 2 does not dominate row 4. (row 2 worse if chose ________)

GAME THEORY

$$
\begin{array}{ll}
 & \text{Sal's} \\
 & \begin{array}{cccc} \text{No special} & \text{Mini} & \text{Medium} & \text{Drink} \end{array} \\
\text{Dino's} \quad \begin{array}{l} \text{No special} \\ \text{Mini} \\ \text{Medium} \\ \text{Drink} \end{array} & \left[\begin{array}{cccc} \cancel{200} & \cancel{-400} & \cancel{-300} & \cancel{-600} \\ 500 & 100 & 200 & 600 \\ \cancel{400} & \cancel{-100} & \cancel{-200} & \cancel{-300} \\ 300 & 0 & 400 & -200 \end{array} \right]
\end{array}
$$

Notice, that from Dino's point of view, no matter what Sal decides to do, Dino always gets a higher payoff from row 2 (minipizza) than from row 1 (no special) because each value in row 2 is larger than the corresponding one in row 1.

→ It would make no sense for Sal to run No Special. Row 2 DOMINIATES row 1, so draw a line through row 1.

Are any other rows dominated?

Row 2 also dominates row 3. Cross out row 3.

Row 2 does not dominate row 4. (row 2 worse if chose medium)

GAME THEORY

$$
\begin{array}{ll|cccc}
 & & & \text{Sal's} & & \\
 & & \text{No special} & \text{Mini} & \text{Medium} & \text{Drink} \\
 & \text{No special} & \cancel{200} & \cancel{-400} & \cancel{-300} & \cancel{-600} \\
 & \text{Mini} & 500 & 100 & 200 & 600 \\
\text{Dino's} & \text{Medium} & \cancel{400} & \cancel{-100} & \cancel{-200} & \cancel{-300} \\
 & \text{Drink} & 300 & 0 & 400 & -200
\end{array}
$$

Now consider Sal's point of view – the column player. Since the column player's payoffs are the opposite of the row player's, a column is dominated if all of its values are larger (not smaller) than another column.

Remember, from the column point of view, smaller values win and larger values lose.

Are there any others?

→ We only look at remaining values. If we only consider the values that have not already been crossed out, eliminate ________________

GAME THEORY

$$
\begin{array}{ll}
 & \text{Sal's} \\
 & \begin{array}{cccc} \text{No special} & \text{Mini} & \text{Medium} & \text{Drink} \end{array} \\
\text{Dino's} \begin{array}{l} \text{No special} \\ \text{Mini} \\ \text{Medium} \\ \text{Drink} \end{array} & \begin{bmatrix} 200 & -400 & -300 & -600 \\ 500 & 100 & 200 & 600 \\ 400 & -100 & -200 & -300 \\ 300 & 0 & 400 & -200 \end{bmatrix}
\end{array}
$$

Now consider Sal's point of view – the column player. Since the column player's payoffs are the opposite of the row player's, a column is dominated if all of its values are larger (not smaller) than another column.

Remember, from the column point of view, smaller values win and larger values lose.

→ All of the values in column 1 are larger that column 2, so Column 1 is dominated. Cross out column 1.

Are there any others?

→ Column 2 dominates column 3 with its remaining values. Notice we only consider the values that have not already been crossed out.

GAME THEORY

$$
\begin{array}{ll}
 & \text{Sal's} \\
 & \begin{array}{cccc} \text{No special} & \text{Mini} & \text{Medium} & \text{Drink} \end{array} \\
\text{Dino's} \begin{array}{l} \text{No special} \\ \text{Mini} \\ \text{Medium} \\ \text{Drink} \end{array} & \begin{bmatrix} 200 & -400 & -300 & -600 \\ 500 & 100 & 200 & 600 \\ 400 & -100 & -200 & -300 \\ 300 & 0 & 400 & -200 \end{bmatrix}
\end{array}
$$

Row Minimums

Column Maximums

Saddle Point

With some of the rows and columns eliminated, the game is easier to examine for maximin and minimax.

The game is ____________________________ with a saddle point of ______.

Dino's best strategy is to offer the ________________________________.
Sal's best strategy is to offer the __________________________________.
Dino will gain about ________ a week over Sal.

GAME THEORY

		Sal's				
		No special	Mini	Medium	Drink	Row Minimums
Dino's	No special	200	−400	−300	−600	
	Mini	500	100	200	600	100 Maximin
	Medium	400	−100	−200	−300	
	Drink	300	0	400	−200	−200
Column Maximums			100 Minimax		600	

Saddle Point

With some of the rows and columns eliminated, the game is easier to examine for maximin and minimax.

The game is strictly determined with a saddle point of 100.

Dino's best strategy is to offer the free minipizza. Sal's best strategy is ALSO to offer the free minipizza. Dino will gain about \$100 a week over Sal.

Practice

1. Each of the following matrices represents a payoff matrix for a game. Determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game. If the game is not strictly determined, explain why.

d. $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -2 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 0 & -6 & 1 \\ -4 & 8 & 2 \\ 6 & 5 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & 2 \\ -1 & -4 & 0 \end{bmatrix}$

2. Use the concept of dominance to solve each of the following games. Give the best row and column strategies and the saddle point of each game.

a.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{bmatrix} 3 & 1 & 7 \\ 0 & 1 & 3 \\ 4 & 3 & 4 \\ 1 & 3 & 6 \end{bmatrix} \end{array}.$$

b.
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} E & F & G \end{array} \\ \begin{bmatrix} 4 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 5 \\ 3 & 2 & 4 \end{bmatrix} \end{array}$$

Practice Answers

1. Each of the following matrices represents a payoff matrix for a game. Determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game. If the game is not strictly determined, explain why.

d. $$\begin{array}{cc} \begin{bmatrix} 0 & 1 & 2 \\ 3 & -2 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ -2 \end{array} \\ \begin{array}{ccc} 3 & 1 & 2 \end{array} & \end{array}$$

Not a Strictly Determined Game because maximin does not equal minimax.

e. $$\begin{array}{cc} \begin{bmatrix} 0 & -6 & 1 \\ -4 & 8 & 2 \\ 6 & 5 & 4 \end{bmatrix} & \begin{array}{c} -6 \\ -4 \\ 4 \end{array} \\ \begin{array}{ccc} 6 & 8 & 4 \end{array} & \end{array}$$

Saddle Point at 4. Best strategies are row 3 and column 3.

f. $$\begin{array}{cc} \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & 2 \\ -1 & -4 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ -3 \\ -4 \end{array} \\ \begin{array}{ccc} 0 & 3 & 2 \end{array} & \end{array}$$

Saddle Point at 0. Best strategies are row 1 and column 1.

Practice Answers

2. Use the concept of dominance to solve each of the following games. Give the best row and column strategies and the saddle point of each game.

a.

$$\begin{array}{c|ccc|c} & E & F & G & \\ \hline A & 3 & 1 & 7 & 1 \\ B & 0 & 1 & 3 & 0 \\ C & 4 & 3 & 4 & 3 \\ D & 1 & 3 & 6 & 1 \\ \hline & 4 & 3 & 7 & \end{array}$$

b.

$$\begin{array}{c|ccc|c} & E & F & G & \\ \hline A & 4 & -1 & -2 & -2 \\ B & 0 & 1 & 1 & 0 \\ C & 0 & -2 & 5 & -2 \\ D & 3 & 2 & 4 & 2 \\ \hline & 4 & 2 & 5 & \end{array}$$

Saddle Point at 3.
Best strategy for the Row Player is option C, and for the Column Player is option F.

Saddle point at 2.
Best strategy for the Row Player is option D, and for the Column Player is option F.

Today's Classwork and Homework

Classwork

- Packet p.12 #1 & 2

Homework

- Packet p. 10
- And Packet p. 8