Matrices & Game Theory Unit

Day 6

Markov Chains

Warm Up:

Put Phones OFF and in the pockets!

The following data is for a certain species of rodent.

Age(months)	0 - 3	3 - 6	6 - 9	9 - 12	12 - 15	15 - 18
Birthrate	0	0.7	1.3	1.1	0.5	0
Survival Rate	0.6	0.9	0.7	0.7	0.5	0
Initial Population	15	12	13	9	6	0

1. Find the population distribution and total population after 21 months.
2. Find the population distribution and total population after 6 cycles.
3. Find the Long Term Growth Rate.
4. By hand, calculate the first population distribution.

HW Questions?

From HW on Packet p3 #9 and Packet p4-5

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Notes Day 6

Markov Chains

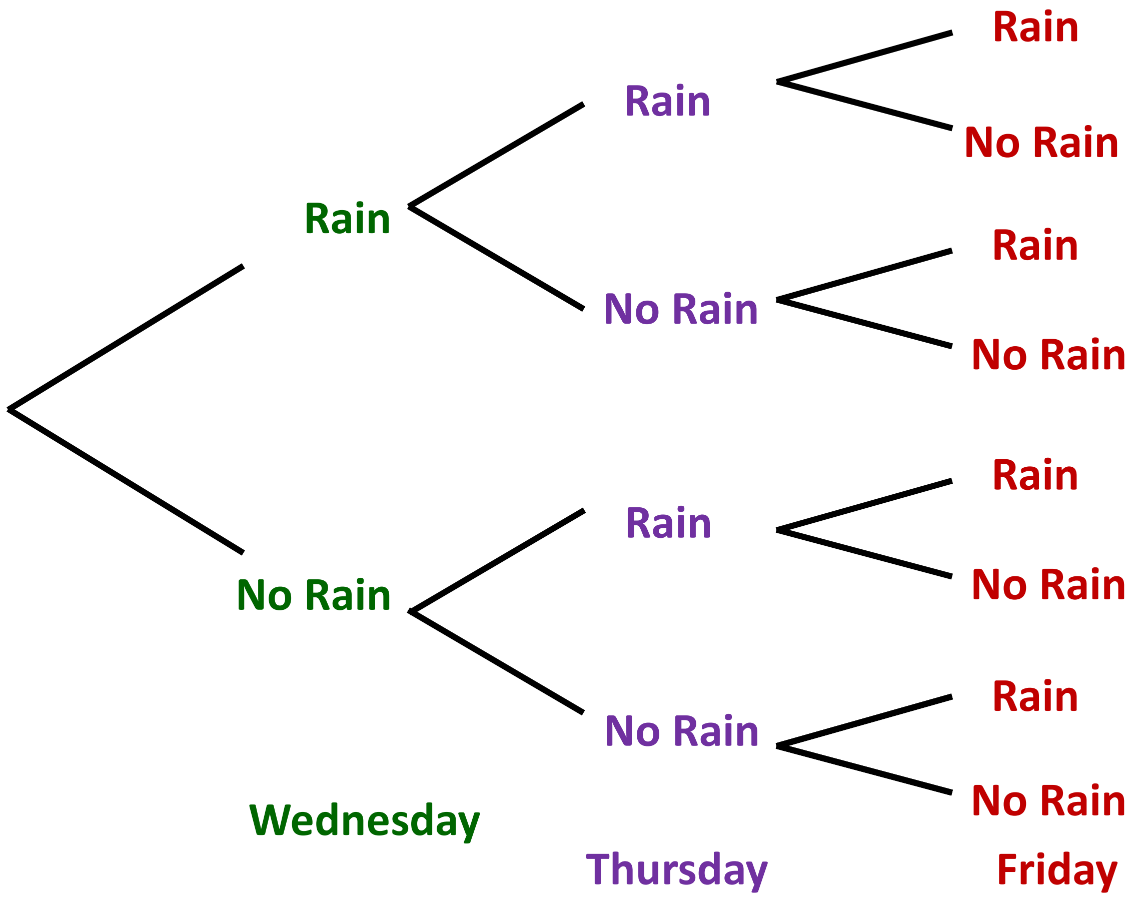
A **Markov Chain** is a process that arises naturally in problems that involve a finite number of events or states that change over time.

The student council at Central High is planning their annual all school May Day games for Friday. They've called the weather forecaster at a local television station and found that:

If it rains on a given day in May the probability of rain the next day is 70%.

If it does not rain on a given day then the probability of rain the next day is 40%.

What is the probability that it will rain on Friday if it is raining on Wednesday?



We can represent this graphically with a tree.

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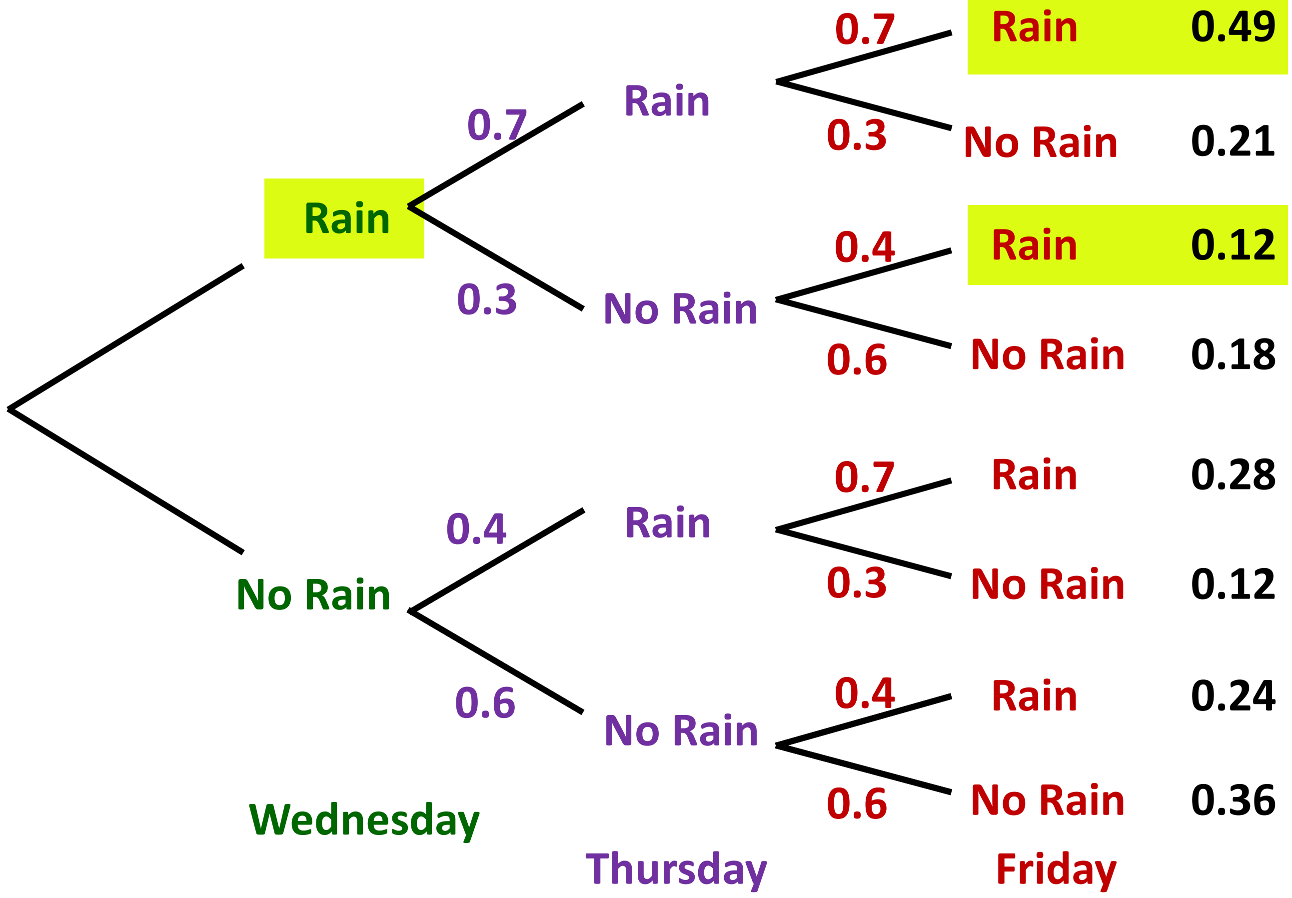


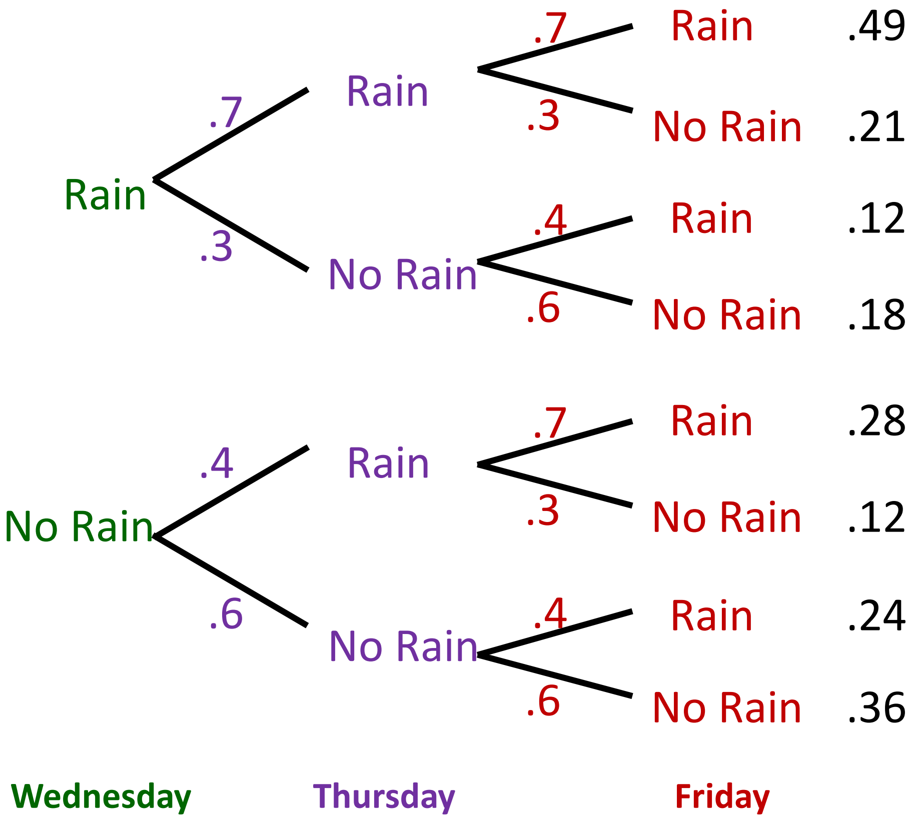
We can represent this graphically with a tree.

$0.49 + 0.12 = 0.61$

61% chance that it will rain on Friday after it rained on Wednesday.

Markov Chains

What if we needed to predict the probability that it will rain much farther out?



We could continue this tree for more and more days.

But, there's got to be a better way.

We can put the probabilities of rain and no rain into a matrix called a:

TRANSITION MATRIX

	Rain Tomorrow	No Rain Tomorrow
Rain Today	.7	.3
No Rain Today	.4	.6

$$\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

We also need an INITIAL DISTRIBUTION matrix.

	Yes	No
Rain on Wednesday	1	0

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Markov Chains

Initial Distribution

	Yes	No
Rain on Wednesday	1	0

Transition Matrix

	Rain Tomorrow	No Rain Tomorrow
Rain Today	.7	.3
No Rain Today	.4	.6

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \bullet \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} .7 & .3 \end{bmatrix}$$

Probabilities of rain and no rain on Thursday.

$$\begin{bmatrix} .7 & .3 \end{bmatrix} \bullet \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} .61 & .39 \end{bmatrix}$$

Probabilities of rain and no rain on Friday.

61% probability of rain on Friday when it rained on Wednesday.

Interesting… that's what we came up with when we did the tree.

EXAMPLE 2

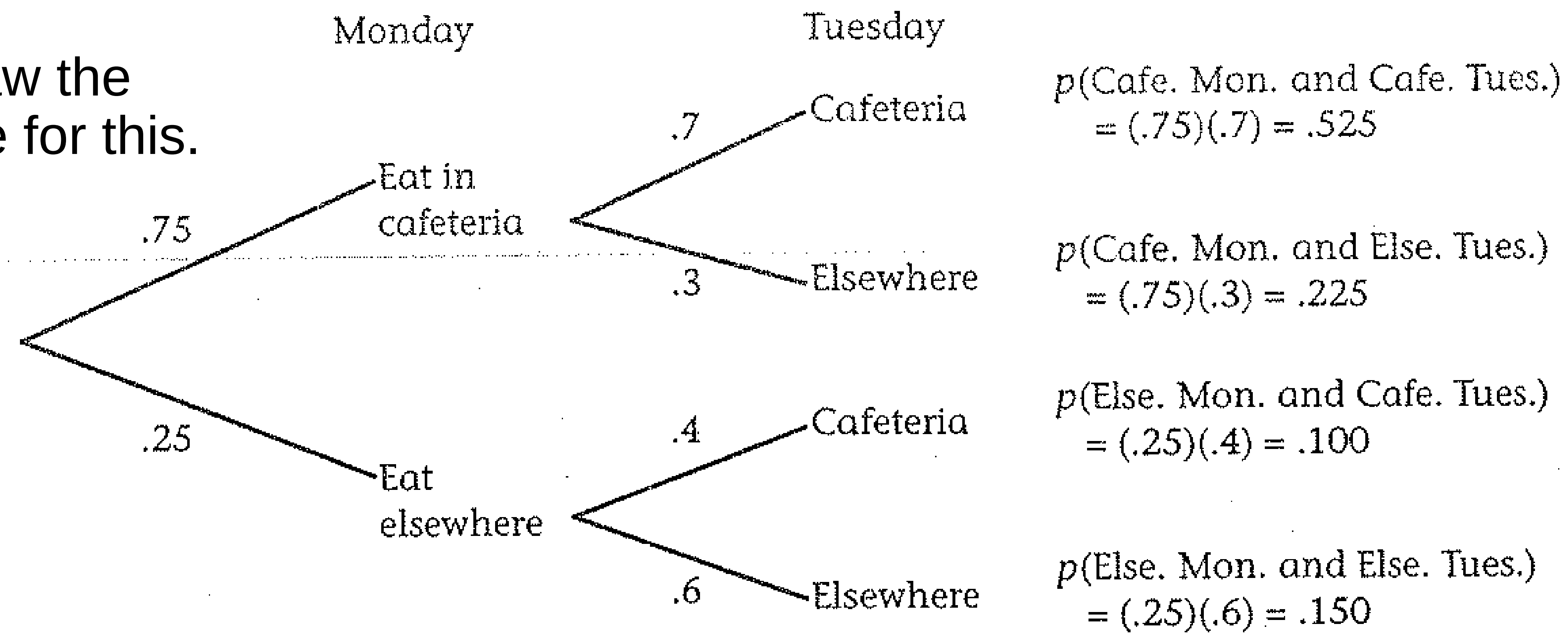
Consider the Lincoln High cafeteria example starting on p.363 of the book.

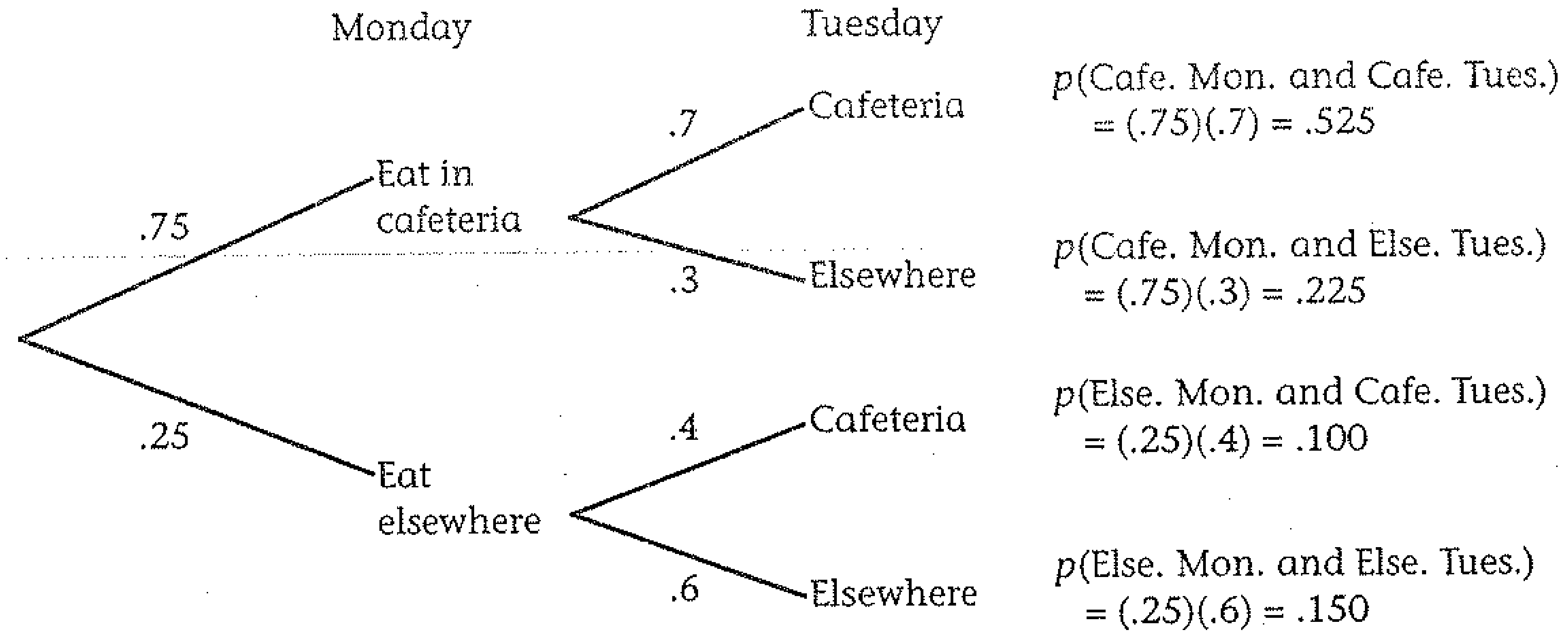
The director of food services wants to predict how many students to expect in the cafeteria in the long run. Students can either eat on campus or elsewhere.

- If a student eats in the cafeteria on a given day, there is a 70% chance they will eat there again and 30% they will not.
- If a student does not eat in the cafeteria on a given day, the probability that she will eat in the cafeteria is 40%, and 60% that she will eat elsewhere.
- On Monday, 75% of the students ate in the cafeteria, 25% did not. What can we expect Tuesday?

- Draw the tree for this example. Start with Monday.

- If eats in the cafeteria, there is 70% chance eat there again and 30% they will not.
- If does not eat in the cafeteria, then probability that eat in the cafeteria is 40%, and 60% that she will eat elsewhere.
- On Monday, 75% ate in cafeteria, 25% did not. What expect Tuesday?
- Draw the tree for this.





The Monday student data are called the **initial distribution** of the student body and can be represented by a row (or **initial state**) vector, $\mathbf{D_0}$, where

$$D_0 = \begin{matrix} C & E \\ [.75 & .25] \end{matrix}$$

C = eats in the cafeteria
E = eats elsewhere.

Movement from one state to another is often called **a transition**, so the data about how students choose to eat from one day to the next is written in a matrix called a **transition matrix, T,** where

$$T = \begin{matrix} & \begin{matrix} C & E \end{matrix} \\ \begin{matrix} C \\ E \end{matrix} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{matrix}.$$

Be consistent in your ordering C then E!

Entries in a transition matrix must be probabilities, values between 0 and 1 inclusive. Also a transition matrix is a square and the sum of the probabilities in any row is 1.

Now calculate the product of matrix D_0 and matrix T:

$$D_1 = D_0T = [.75 \quad .25]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.75(.7) + .25(.4) \quad .75(.3) + .25(.6)]$$

$$= [.625 \quad .375].$$

To see what happens on Wednesday, it is only necessary to repeat the process using D_1 in place of D_0:

$$D_2 = D_1T = [.625 \quad .375]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.625(.7) + .375(.4) \quad .625(.3) + .375(.6)]$$

$$= [.5875 \quad .4125].$$

$D_2 = D_1T$, but $D_1 = D_0T$, so by substitution, $D_2 = (D_0T)(T)$.

Because matrix multiplication is associative,

$$D_2 = (D_0T)(T) = D_0(T^2).$$

Markov Chains

So, the general formula for a Markov Chain is :

$$D_n = D_0 T^n$$

D_0 = Initial Distribution

T = Transition Matrix

n = Iteration Number

Markov Chains

Classwork Packet p. 6-7

Let’s Start p. 6 #1 together

Markov Chains

The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day.

Represent this information in a transition matrix. Hint: Draw a tree

T =

	Rain	Nice	Snow
Rain			
Nice			
Snow			

What is the long-term expectation of weather in Oz ?

Since no D_0 information was given, we won't use the full D_n formula. Instead, explore T to different powers.

Markov Chains

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$$T = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

Represent this information in a transition matrix. Hint: Draw a tree

What is the long-term expectation of weather in Oz ?

Since no D_0 information was given, we won't use the full D_n formula. Instead, explore T to different powers.

$T^{20} =$ $T^{21} =$ $T^{22} =$ … $T^{30} =$ $T^{31} =$ $T^{32} =$

$$\begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{bmatrix} .4 & .2 & .4 \\ .4 & .2 & .4 \\ .4 & .2 & .4 \end{bmatrix} \end{array}$$

→ Long term, in Oz they can expect rain 40%, nice weather 20%, and snow 40% of the time.

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