Game Theory Continued
Day 6

Non-Strictly Determined

Section 7.5

Warm Up

There are 50 students in a first period band class. Twelve were early to class, 34 were on time, and 4 were late. Fifteen percent of those who were early to first period are on time to their next class and 5% are late. Of the students who were on time to first period, 10% are late to the next class while the others were on time. Of the students that were late to first period, 12% are late to the next class and 40% were early.

a. What is the initial matrix for the class of students?

b. What is the transition matrix for the class of students?

c. Approximately how many students will be late after 6 classes?

Warm Up

There are 50 students in a first period band class. 12 were early to class, 34 were on time, and 4 were late. 15% of those who were early to first period are on time to their next class and 5% are late. Of the students who were on time to first period, 10% are late to the next class while the others were on time. Of the students that were late to first period, 12% are late to the next class and 40% were early.

a. What is the initial matrix for the class of students?

$$D_0 = \begin{bmatrix} 12 & 34 & 4 \end{bmatrix}$$

b. What is the transition matrix for the class of students?

$$T = \begin{bmatrix} .8 & .15 & .05 \\ 0 & .9 & .10 \\ .4 & .48 & .12 \end{bmatrix}$$

c. Approximately how many students will be late after 6 classes? 4.6 students

Warm Up

- Check you HW answers with people at your table, packet p.11. Volunteers to put answers on the board?

- Check your quiz corrections with people at your table.

GAME THEORY

Section 7.5 is for games that are NOT Strictly Determined.

Remember, GAME THEORY is really just DECISION-MAKING.

GAME THEORY

Coin Game – Two Players (Player-R & Player-C)

Both players simultaneously display a coin.
- This is not a random flip. The player chooses which side to display.

If both players display heads, then Player-R wins 4¢ from Player-C.

If both players display tails, then Player-R wins 1¢ from Player-C.

If Player-R displays a head and Player-C displays a tail, then Player-R pays Player-C 2¢.

If Player-R displays a tail and Player-C displays a head, then Player-R pays Player-C 3¢.

What is the best STRATEGY for each player ?

Make the PAYOFF MATRIX.

GAME THEORY

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Our text describes Sol and Tina playing our coin game. Sol is Player-R and Tina is Player-C.

If you figured out the game, you should have found that this game is NOT Strictly Determined.

But, is it FAIR?

Is there a better strategy for Sol or Tina?

Show the Payoff Matrix.

$$\begin{array}{ccc} & & \text{Tina} \\ & & \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \text{Sol} & \begin{array}{c} \text{Heads} \\ \text{Tails} \end{array} & \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \end{array}$$

GAME THEORY

Find the Maximin and Minimax.

$$
\begin{array}{llccc}
 & & \text{Tina} & & \text{Row} \\
 & & \begin{array}{cc}\text{Heads} & \text{Tails}\end{array} & & \text{Minimums} \\
\text{Sol} & \begin{array}{l}\text{Heads} \\ \text{Tails}\end{array} & \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} & & \begin{array}{c} -2 \\ -3 \end{array} \\
\text{Column Maximums} & & \begin{array}{cc} 4 & 1 \end{array} & &
\end{array}
$$

Maximin: -2

Minimax: 1

Do you know how to explain what the maximin of -2 means in this problem?

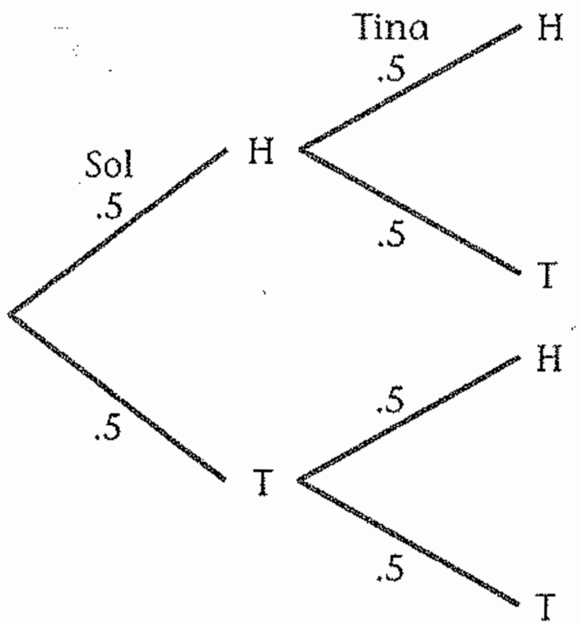
The Maximin is –2 and the Minimax is 1. Since they are not the same, there is no saddle point and the game is <u>not Strictly Determined</u>.

The best strategy for either player is to display a mix of heads and tails to keep the other player guessing.

Would it be best to just flip the coin so that a player gets about half heads and half tails? Would it be better to display heads more than tails?

GAME THEORY

Consider what would happen if Sol and Tina each decide to just flip their coins.



p(both Sol and Tina show heads) = .5(.5) = .25

p(Sol shows heads and Tina shows tails) = .5(.5) = .25

p(Sol shows tails and Tina shows heads) = .5(.5) = .25

p(both Sol and Tina show tails) = .5(.5) = .25

The probability distribution for Sol's winnings for this case are shown in this probability table.

Outcome	HH	HT	TH	TT
Probability	.25	.25	.25	.25
Amount won	4	−2	−3	1

Find the expected payoff of the game for Sol.

GAME THEORY

Outcome	HH	HT	TH	TT
Probability	.25	.25	.25	.25
Amount won	4	−2	−3	1

Sol's expected payoff is . . .

$$\mu = .25(4) + .25(-2) + .25(-3) + .25(1) =$$

$$1.00 \quad -.50 \quad -.75 \quad +.25 \quad = 0$$

Tina's payoffs are the opposite of Sol's, so her expected payoff is . . .

$$\mu = .25(-4) + .25(2) + .25(3) + .25(-1) =$$

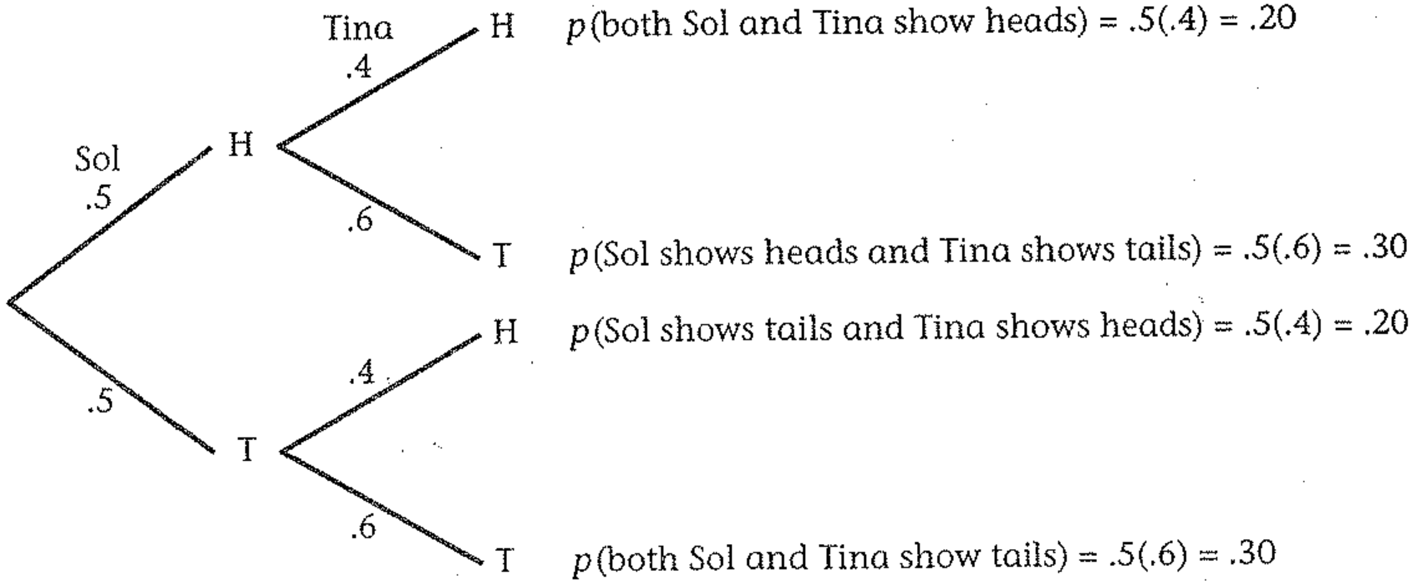
$$-1.00 \quad +.50 \quad +.75 \quad -.25 \quad = 0$$

With this strategy the game is FAIR because both players' expectations are equal.

Well, being fair seems nice, but I'd rather have an advantage. Could one player choose to do something other than 50% heads, 50% tails to gain an advantage ?

GAME THEORY

Consider what would happen if Tina decides to play 40% heads and 60% tails. Make a probability tree.



Now Sol's probability distribution looks like this.

Outcome	HH	HT	TH	TT
Probability	.2	.3	.2	.3
Amount won	4	−2	−3	1

Now find Sol's expected value.

GAME THEORY

Outcome	HH	HT	TH	TT
Probability	.2	.3	.2	.3
Amount won	4	−2	−3	1

Sol's expected payoff now is . . .

$$\mu = .2(4) + .3(-2) + .2(-3) + .3(1) =$$

$$.8 \quad -.6 \quad -.6 \quad +.3 \quad = -.1$$

Tina's payoffs are the opposite of Sol's, so her expected payoff is . . .

$$\mu = .2(-4) + .3(2) + .2(3) + .3(-1) =$$

$$-.8 \quad +.6 \quad +.6 \quad -.3 \quad = +.1$$

This means Sol will lose 0.1 pennies per play or one penny for every 10 plays. Tina has an advantage and the game is no longer fair.

GAME THEORY

Outcome	HH	HT	TH	TT
Probability	.2	.3	.2	.3
Amount won	4	−2	−3	1

Of course, there's a matrix technique to figure all this out. Here's how we can figure out Sol, the row player's, expected payoff.

$$\begin{bmatrix} .5 & .5 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} -.1 \end{bmatrix}$$

Row player's expected payoff per play.

Sol's percentage of head and tails in a row matrix.

Payoff matrix.

Tina's percentage of head and tails in a column matrix.

Well, Sol's not going to play 50/50 all the time, and Tina's going to vary her strategy, as well. How can each player maximize their advantage (their long-term expected payoff)?

GAME THEORY

We need a way to figure out each player's Best Strategy.

Here is the general technique for how we do this.

Sol's Best Strategy. **p = percentage of heads**

(1 – p) = percentage of tails.

$$\begin{bmatrix} p & (1-p) \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} p(4)+(1-p)(-3) & p(-2)+(1-p)(1) \end{bmatrix}$$

Simplify.

$$\begin{bmatrix} 4p-3+3p & -2p+1-p \end{bmatrix}$$

$$\begin{bmatrix} 7p-3 & -3p+1 \end{bmatrix}$$

Set these equal. Solve for p.

$$7p-3=-3p+1$$

$$10p=4$$

$$p=\frac{4}{10}=.4$$

$$1-p=1-.4=.6$$

Sol's best strategy is to play heads 40% and tails 60%.

GAME THEORY

Well, Tina's pretty smart. She can find her best strategy, too.

Here's how we find the best strategy for the column player.

Tina's Best Strategy. **q = percentage of heads**
(1 – q) = percentage of tails.

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \bullet \begin{bmatrix} q \\ (1-q) \end{bmatrix} = \begin{bmatrix} 4q + (-2)(1-q) \\ -3q + (1)(1-q) \end{bmatrix}$$

Simplify.

$$\begin{bmatrix} 4q - 2 + 2q \\ -3q + 1 - q \end{bmatrix}$$

$$\begin{bmatrix} 6q - 2 \\ -4q + 1 \end{bmatrix}$$

Set these equal.
Solve for q.

$$6q - 2 = -4q + 1$$

$$10q = 3$$

$$q = \frac{3}{10} = .3$$

$$1 - q = 1 - .3 = .7$$

Tina's best strategy is to play heads 30% and tails 70%.

GAME THEORY

These two matrix multiplication processes are the techniques you will use to find the best strategies for two opposing players.

We will not learn techniques for games that have more than two strategies. That's just slightly above the scope of our course.

If both Sol and Tina use their best strategies, what is the expected outcome of the game?

$$\begin{bmatrix} .4 & .6 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} -.2 \end{bmatrix}$$

Sol's Best Strategy.

Payoff matrix.

Tina's Best Strategy.

Sol's Expected Outcome.

Game Theory

As a 2nd example, let's work through Section 7.5 #6 together.

In a game known as Two-Finger Morra, two players simultaneously hold up either one or two fingers. If they hold up the same number of fingers, player 1 will win the sum (in pennies) of the digits from player 2. If they hold up different numbers, then player 2 will win the sum from player 1. Write the payoff matrix for this game. Find the best strategy for each player and the expectation for the row player. Is this a fair game? Explain your answer.

$$\begin{array}{cc} & \text{Player 2} \\ \text{Player 1} & \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \end{array} \end{array}$$

$$\begin{bmatrix} p & 1-p \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2p-3+3p & -3p+4-4p \end{bmatrix}$$

$$2p-3+3p = -3p+4-4p$$

$$5p-3 = -7p+4$$

$$12p = 7$$

$$p = \frac{7}{12}$$

$$1-p = \frac{5}{12}$$

$$\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 2q-3+3q \\ -3q+4-4q \end{bmatrix}$$

$$2q-3+3q = -3q+4-4q$$

$$5q-3 = -7q+4$$

$$12q = 7$$

$$q = \frac{7}{12}$$

$$1-q = \frac{5}{12}$$

Game Theory

In a game known as Two-Finger Morra, two players simultaneously hold up either one or two fingers. If they hold up the same number of fingers, player 1 will win the sum (in pennies) of the digits from player 2. If they hold up different numbers, then player 2 will win the sum from player 1. Write the payoff matrix for this game. Find the best strategy for each player and the expectation for the row player. Is this a fair game? Explain your answer.

$$\begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} \end{bmatrix} \approx [-.0833]$$

The row player can expect to lose about 8 cents in every 100 plays or one penny in every twelve plays.

Practice

Suppose that Sol and Tina change their game so that the payoffs to Sol are

$$\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

a. Use the row matrix $\begin{bmatrix} p & 1-p \end{bmatrix}$ to find Sol's best strategy for this game.

b. Use the column matrix $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ to find Tina's best strategy for this game.

c. Set up a tree diagram to compute the probabilities of each of the four outcomes for this game.

d. Prepare a probability distribution chart for Sol's winnings.

e. Find Sol's expectation for this game.

Answer

$$\begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \end{bmatrix} \approx [-.125]$$

a. Sol will play heads 3 of the 8 times and tails 5 of the 8 times.
b. Tina will play heads 3 of the 8 times and tails 5 of the 8 times.
e. So Sol is expected to lose 1 penny for every 8 games he plays.

Classwork/Homework
Packet p. 10-12

- Packet p. 10 #8, eliminate 1st row and 1st column using dominance method.
- Packet page 11 you should have already completed previously in class.
- Packet p.12 #6 we will do tomorrow together on the Review Day