Matrices Unit Day 5

Growth Rate with Leslie Matrices and Practice with Matrix Applications

Warm Up

1) Write a general matrix equation (write step in the calculator) that would find the distribution of a population for the 9th cycle.

Use correct notation!

3) Use a matrix operation(s) to calculate a 30% discount on the ticket prices shown in the table. Label your rows and columns.

Fall Park Tickets	Scaro-winds	Howl-O-Scream
Adults	$60	$77
Kids	$50	$67

4) Suppose you are planning a family vacation during which you will visit each of the parks twice. You will visit the park during Thanksgiving Break, a peak time, so the cost will be regular price. Your family has 2 adults and 3 children. Use matrix operations to find the total cost to expect when booking tickets for each park.

Warm Up Answers

Use correct notation!

1) Write a general matrix equation (write step in the calculator) that would find the distribution of a population for the 9th cycle.

$P_9 = P_0 \bullet L^9$

2) Write a general matrix equation that would find the total population for the 9th cycle.

Total Pop of $P_9 = P_0 \bullet L^9 \bullet C$

3) Use a matrix operation(s) to calculate a 30% discount on the ticket prices shown. Label your rows and columns.

Fall Park Tickets	Scaro-winds	Howl-O-Scream
Adults	$60	$77
Kids	$50	$67

Use Scalar Multiplication **OR** Matrix Subtraction with Scalar Mult.

$$0.70 \bullet \begin{matrix} & \text{Scaro-winds} & \text{Howl-o Scream} \\ \text{Adults} & \$60 & \$77 \\ \text{Kids} & \$50 & \$67 \end{matrix} =$$

$$\begin{matrix} & \text{Scaro-winds} & \text{Howl-o Scream} \\ \text{Adults} & \$42 & \$53.90 \\ \text{Kids} & \$35 & \$46.90 \end{matrix}$$

Warm Up ANSWERS

Fall Park Tickets	Scaro-winds	Howl-O-Scream
Adults	$60	$77
Kids	$50	$67

4) Suppose you are planning a family vacation during which you will visit each of the parks twice. You will visit the park during Thanksgiving Break, a peak time, so the cost will be regular price. Your family has 2 adults and 3 children. Use matrix operations to find the total cost to expect when booking tickets for each park.

Use Scalar Multiplication **AND** Matrix Multiplication

$$2 \bullet \overset{\begin{matrix} Adults & Kids \end{matrix}}{\begin{bmatrix} 2 & 3 \end{bmatrix}} \bullet \begin{matrix} Adults \\ Kids \end{matrix} \overset{\begin{matrix} Scaro-winds & Howl-o\ Scream \end{matrix}}{\begin{bmatrix} \$60 & \$77 \\ \$50 & \$67 \end{bmatrix}} = \text{Total Cost} \overset{\begin{matrix} Scaro-winds & Howl-o\ Scream \end{matrix}}{\begin{bmatrix} \$540 & \$710 \end{bmatrix}}$$

Homework Questions?

Tonight’s Homework

- Packet p. 3 #9
- Finish Packet p. 4-5

Population Growth: The Leslie Model (Part 2) Section 3.5

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{juveniles} \\ \text{young adults} \\ \text{adults} \\ \text{old adults} \end{bmatrix}$$

Remember…Leslie Model

We can combine our birth rates matrix and our survival rates matrices into a single matrix we will call the LESLIE MATRIX (L)

$$L = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The birth rates are in the first column.

The survival rates are in the Super Diagonal.

The Leslie Matrix MUST ALWAYS be a Square Matrix!!

Remember….
Leslie Model helps calculate population distributions

Find the population distribution, P_n, after n cycles with

$$P_n = P_0 L^n$$

Where P_0 is the initial population

Examples:

To find population distribution after 2 cycles use

$$P_2 = P_0 L^2$$

To find population distribution after 8 cycles use

$$P_8 = P_0 L^8$$

Remember….
Leslie Model ALSO helps calculate population totals

Find the total population after n cycles with

$$\text{total } P_n = P_0 \bullet L^n \bullet C$$

Where P_0 is the initial population

And C is the column matrix

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise #2

Suppose the rats start dying off from overcrowding when the total female population for a colony reaches 250. Find how long it will take for this to happen when the initial population distribution is:

$a)\begin{bmatrix} 18 & 9 & 7 & 0 & 0 & 0 \end{bmatrix}$

a) 60 cycles

$b)\begin{bmatrix} 35 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

You Try!

b) 69 cycles

Age (months)	Birthrate	Survival Rate	Initial Population of Rats
0-3	0	0.6	15
3-6	0.3	0.9	9
6-9	0.8	0.9	13
9-12	0.7	0.8	5
12-15	0.4	0.6	0
15-18	0	0	0

Growth Rates Between Cycles

STILL working with the rats.
Consider the table below.

$$P_0 = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix}$$

Cycle	Total Population	Growth Rate
Original	42	
1	49.4	17.6%
2	56.08	13.5%
3	57.40	2.4%
4	**56.65**	**–1.31%**
5	**59.35**	**4.77%**
6	**61.76**	**4.06%**

The general formula for finding the growth rate between cycles is:

$$G.R. = \frac{(new - previous)}{previous}$$

To find the growth rate between the initial population and the first cycle we would do the following calculation.

$$\frac{(49.4 - 42)}{42} \approx .176$$

Notice the growth rates appear to decline then increase again.

Growth Rates Between Cycles

Find the growth rates between P_{19} and P_{20}, between P_{20} and P_{21}, between P_{25} and P_{26}, between P_{26} and P_{27}

$$P_0 = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix}$$

Cycle	Total Population	Growth Rate
19	90.627	
20	93.384	**3.04%**
21	**96.230**	**3.05%**
25	**108.488**	
26	**111.789**	**3.04%**
27	**115.191**	**3.04%**

One characteristic of the Leslie Model is that the growth does stabilize at a rate called the *<u>LONG-TERM GROWTH RATE</u>* of the population.

The long-term growth rate for this rat population is 3.04% or .0304.

Exercise #3

a) Find the long-term growth rate of the total population for each of the initial population distributions in #2 (From earlier today.)

$$a)\begin{bmatrix}18 & 9 & 7 & 0 & 0 & 0\end{bmatrix}$$

a) 3.04%

$$b)\begin{bmatrix}35 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

b) 3.04%

b) How does the initial population distribution seem to affect the long-term growth rate?

The initial population distribution does NOT seem to affect the long term growth rate.

Leslie Model – A Special Case

A special case that P.H. Leslie studied involved creatures that reproduce only in one phase of their life cycle.

Consider the scenario posed below. This certain kind of bug lives only 3 weeks. 50% of the bugs survive from the first week of life to the second. 70% of those who make it to the second week also survive into the third. No bugs live beyond three weeks. On average 6 newborn bugs are produced by those bugs who make it to the third week. Create a table from info.

Use this description to construct a Leslie Matrix for the life cycle of this bug.

Weeks	Birth Rate	Survival Rate
1	0	0.50
2	0	0.70
3	6	0

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Special Case Practice

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Using the same bugs we just discussed…
A group of five 3-week old female bugs move into your basement.

What is P_0 ? $$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

How long will it be before the bug population passes 1,000 bugs ?

Answer on the next slide…

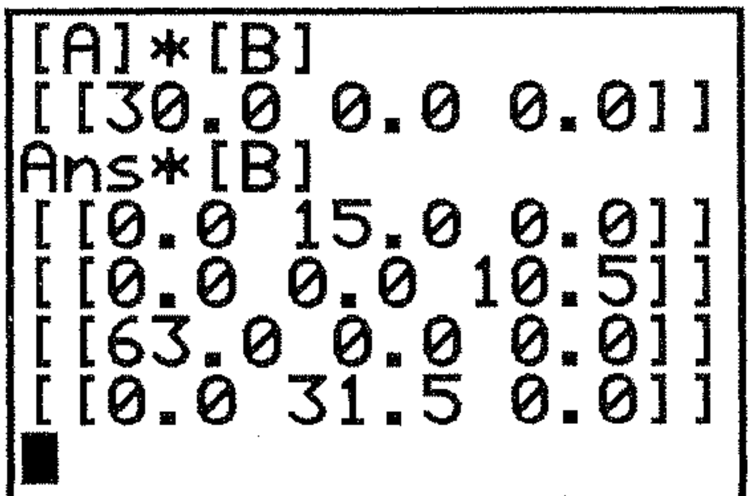
Leslie Model – A Special Case

How long before there are at least 1,000 female bugs in the basement?

$$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Note: To use the iteration function on the graphing calculator:



Press [A] [x] [B] and press [ENTER].

Press [x] [B] and press [ENTER].

Continue to press [ENTER] while you count the cycles.

Use this iteration function to observe the pattern. Could you put this information into a table ?

There will be 1,225 female bugs living in the basement after 16 weeks.

Leslie Model – A Special Case

Make a table of the population distributions for cycles P_{22} through P_{30}.

$$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Cycle	One Week Old	Two Weeks Old	Three Weeks
0	0	0	5
22	5,403.3	0	0
23	0	2,701.6	0
24	0	0	1,891.1
25	11,346.9	0	0
26	0	5,673.4	0
27	0	0	3,971.4
28	23,828.4	0	0
29	0	11,914.2	0
30	0	0	8,339.9

a) Do you observe a pattern?

What about the growth rate? How should we calculate it?

b) Examine the population change from :

P_{22} to P_{25} ,
P_{23} to P_{26} ,
P_{24} to P_{27} ,
P_{25} to P_{28} ,
P_{26} to P_{29} ,
P_{27} to P_{30}

Make a conjecture.

The bug population is approximately doubling every 3 weeks! YIKES!

Packet Practice

- Complete the Long-Term Growth Rate problems
  Packet p. 3 #9, p. 4 #14, and p. 5 #5b

Tonight's Homework

- Packet p. 3 #9
- Finish Packet p. 4-5

Tonight's Homework

- Finish Packet p. 2-4

Next slides…

- Moved to Day 5 for Spr ’18 and Fall ‘18

Notes Day 4

Markov Chains

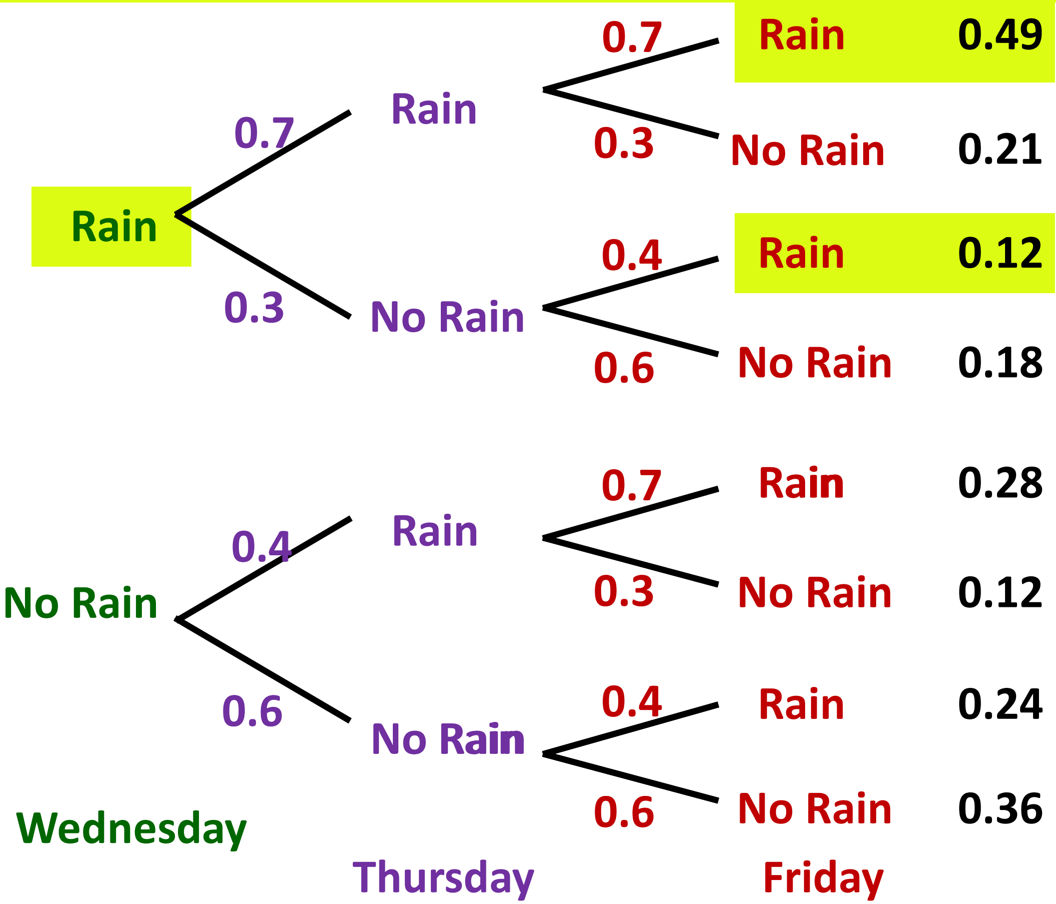
A **Markov Chain** is a process that arises naturally in problems that involve a finite number of events or states that change over time.

The student council at Central High is planning their annual all school May Day games for Friday. They've called the weather forecaster at a local television station and found that:

***If it rains on a given day in May the probability of rain the next day is 70%*.**

If it does not rain on a given day then the probability of rain the next day is 40%.

What is the probability that it will rain on Friday if it is raining on Wednesday?



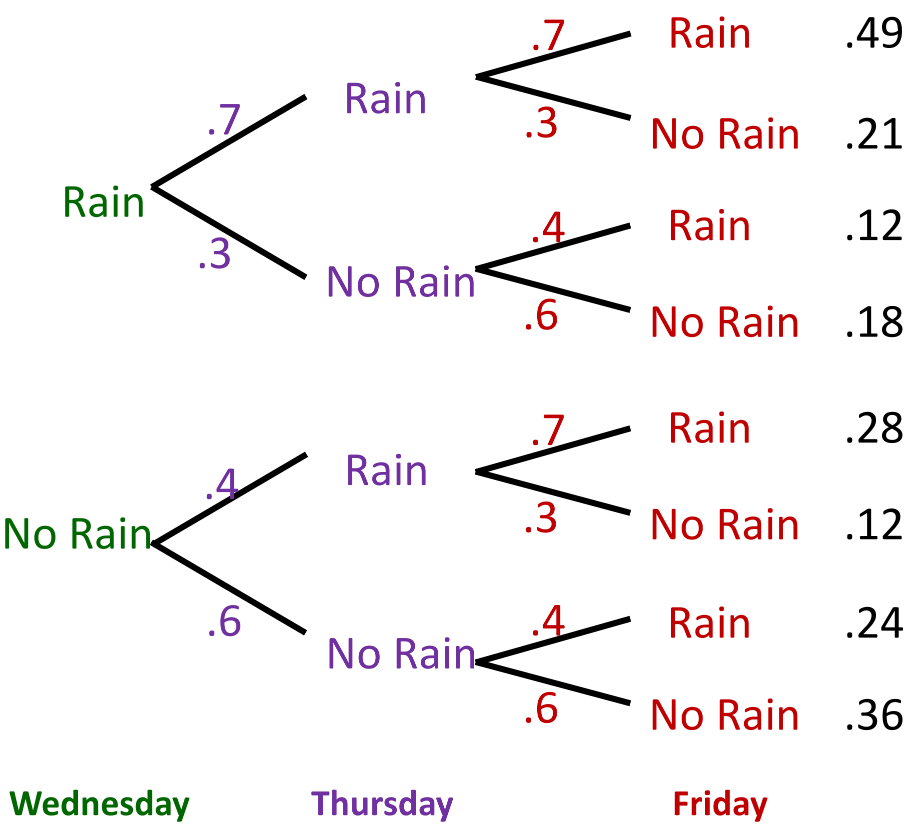
We can represent this graphically with a tree.

0.49 + 0.12 = 0.61

61% chance that it will rain on Friday after it rained on Wednesday.

Markov Chains

What if we needed to predict the probability that it will rain much farther out?



We could continue this tree for more and more days.

But, there's got to be a better way.

We can put the probabilities of rain and no rain into a matrix called a:

TRANSITION MATRIX

	Rain Tomorrow	No Rain Tomorrow
Rain Today	.7	.3
No Rain Today	.4	.6

$$\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

We also need an INITIAL DISTRIBUTION matrix.

	Yes	No
Rain on Wednesday	1	0

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Markov Chains

Initial Distribution

	Yes	No
Rain on Wednesday	1	0

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Transition Matrix

	Rain Tomorrow	No Rain Tomorrow
Rain Today	.7	.3
No Rain Today	.4	.6

$$\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \bullet \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} .7 & .3 \end{bmatrix}$$

Probabilities of rain and no rain on Thursday.

$$\begin{bmatrix} .7 & .3 \end{bmatrix} \bullet \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} .61 & .39 \end{bmatrix}$$

Probabilities of rain and no rain on Friday.

61% probability of rain on Friday when it rained on Wednesday.

Interesting… that's what we came up with when we did the tree.

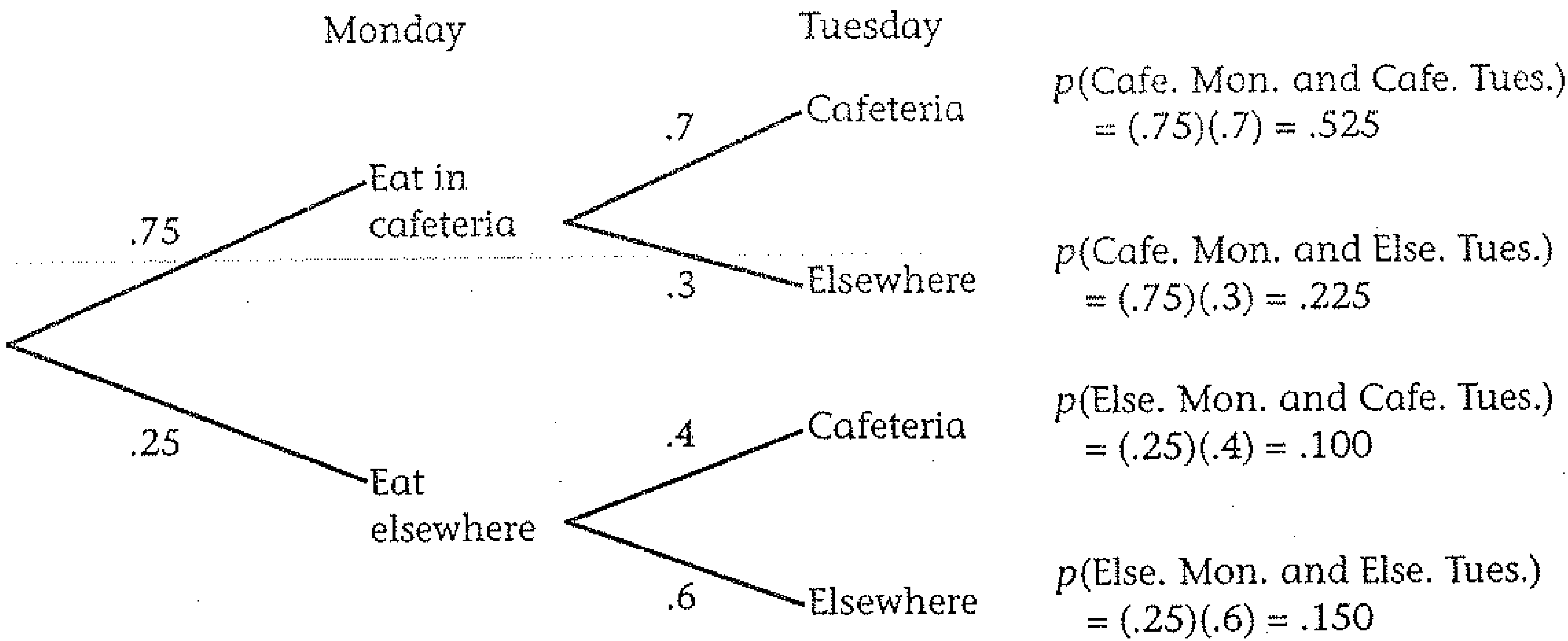
Example 2

The director of food services wants to predict how many students to expect in the cafeteria in the long run. Students can either eat on campus or elsewhere.

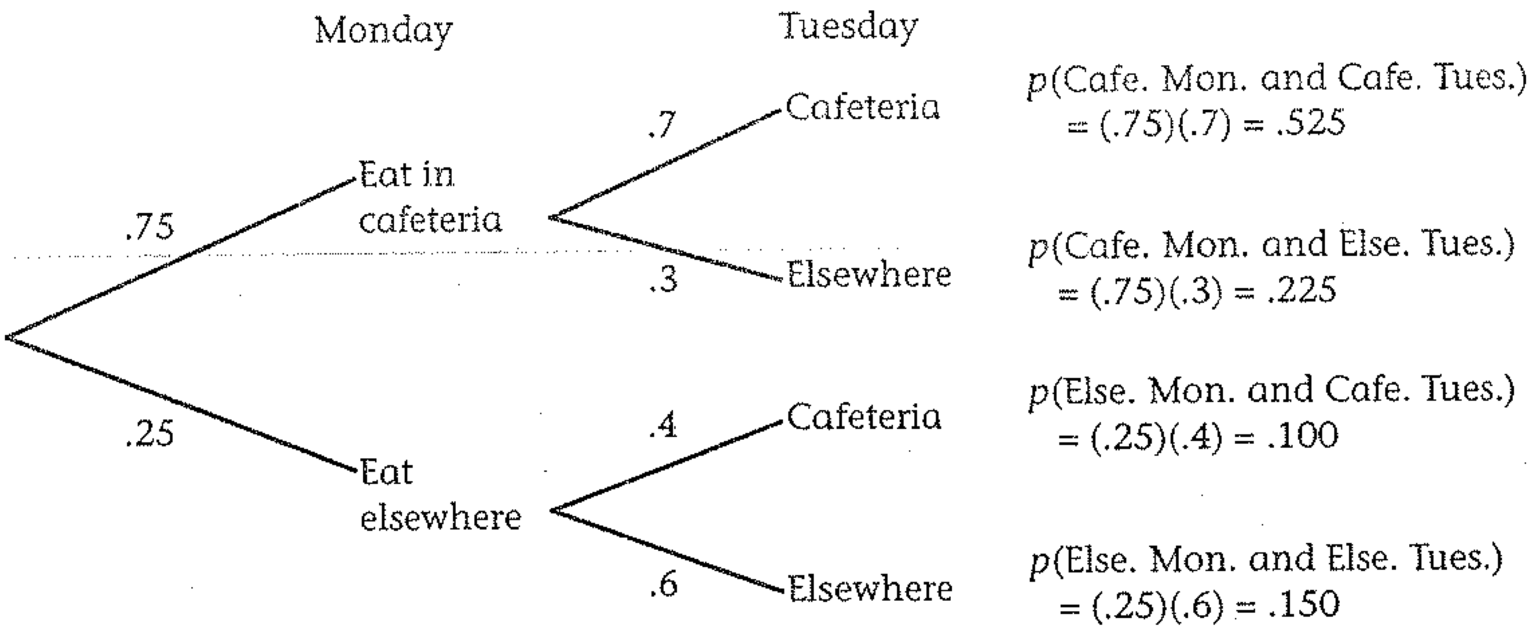
- If a student eat in the cafeteria on a given day, there is a 70% chance they will eat there again and 30% they will not.
- If a student does not eat in the cafeteria on a given day, the probability she will eat in the cafeteria is 40%, 60% she will eat elsewhere.
- On Monday, 75% of the students ate in the cafeteria, 25% did not. What can we expect Tuesday?

- Draw the tree for this example. Start with Monday.

Consider the Lincoln High cafeteria example starting on p.363 on the book.



Consider the Lincoln High cafeteria example starting on p.363 on the book.



The Monday student data are called the **initial distribution** of the student body and can be represented by a row (or **initial-state**) vector, D_0, where

$$D_0 = \begin{matrix} C & E \\ [.75 & .25] \end{matrix}$$

C = eats in the cafeteria
E = eats elsewhere.

Movement from one state to another is often called a **transition**, so the data about how students choose to eat from one day to the next is written in a matrix called a **transition matrix**, T, where

$$T = \begin{matrix} & \begin{matrix} C & E \end{matrix} \\ \begin{matrix} C \\ E \end{matrix} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{matrix}.$$

Be consistent in your ordering C then E!

Entries in a transition matrix must be probabilities, values between 0 and 1 inclusive. Also a transition matrix is a square and the sum of the probabilities in any row is 1.

Now calculate the product of matrix D_0 and matrix T:

$$D_0T = [.75 \quad .25]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.75(.7) + .25(.4) \quad .75(.3) + .25(.6)]$$

$$= [.625 \quad .375].$$

To see what happens on Wednesday, it is only necessary to repeat the process using D_1 in place of D_0:

$$D_1T = [.625 \quad .375]\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} = [.625(.7) + .375(.4) \quad .625(.3) + .375(.6)]$$

$$= [.5875 \quad .4125].$$

$D_2 = D_1T$, but $D_1 = D_0T$, so by substitution, $D_2 = (D_0T)(T)$.

Because matrix multiplication is associative,

$$D_2 = (D_0T)(T) = D_0(T^2).$$

Markov Chains

So, the general formula for a Markov Chain is :

$$D_n = D_0 T^n$$

D_0 = Initial Distribution

T = Transition Matrix

n = Iteration Number

Markov Chains

The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day.

Represent this information in a transition matrix. Hint: Draw a tree

What is the long-term expectation of weather in Oz ?

$$\begin{array}{c} \\ \textbf{Rain} \\ \textbf{Nice} \\ \textbf{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Rain} & \textbf{Nice} & \textbf{Snow} \end{array} \\ \begin{bmatrix} .4 & .2 & .4 \\ .4 & .2 & .4 \\ .4 & .2 & .4 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ \textbf{Rain} \\ \textbf{Nice} \\ \textbf{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Rain} & \textbf{Nice} & \textbf{Snow} \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

Markov Chains

Classwork

Packet p. 5-6

Next up….

- Old slides not used Spring 18

Word Problem Practice

A landlord owns 3 condominiums, a 1-bedroom condo, a 2-bedroom condo, and a 3-bedroom condo. The total rent she receives is $1240. She needs to make repairs on the condos, and it costs 10% of the 1-bedroom condo’s rent for its repairs, 20% of the 2-bedroom for its repairs, and 30% of the 3-bedroom condo’s rent for its repairs. The total repair bill was $276. The 3-bedroom condo’s rent is twice the 1-bedroom condo’s rent. How much is the rent for each of the condos?

Word Problem Practice ANSWERS

A landlord owns 3 condominiums, a 1-bedroom condo, a 2-bedroom condo, and a 3-bedroom condo. The total rent she receives is $1240. She needs to make repairs on the condos, and it costs 10% of the 1-bedroom condo's rent for its repairs, 20% of the 2-bedroom for its repairs, and 30% of the 3-bedroom condo's rent for its repairs. The total repair bill was $276. The 3-bedroom condo's rent is twice the 1-bedroom condo's rent. How much is the rent for each of the condos?

Let x = the 1-bedroom condo's rent,
y = the 2 bedroom condo's rent,
and z = the 3 bedroom condo's rent

$$x + y + z = 1240$$
$$0.10x + 0.20y + 0.30z = 276$$
$$z = 2x$$

$280 is the 1-bedroom condo's rent
$400 is the 2 bedroom condo's rent
$560 is the 3 bedroom condo's rent

Practice~ Day 4 Answers

4) Solve for the variables.

$$\begin{bmatrix} 3x-7 & -8 \\ 23 & 3r+x \end{bmatrix} = \begin{bmatrix} 20 & 12-m \\ y^3+15 & 12 \end{bmatrix}$$

x= 9 **m= 20**

y= 2 **r= 1**

Tonight’s Homework

- Finish Quiz Review Handout

Tonight's Homework

- Finish Quiz Review Handout

- Quiz Tomorrow is on
  - Matrix Operations by hand
  - Matrix Applications (word problems)
  - Solving System of Equations with Matrices
  - Leslie Matrices – including today's stuff about long term growth rate
  - NOT on Markov Chains