ICM Matrices Unit 2
Day 4

Leslie Matrices

Warm Up Day 4

1. A chain of hardware stores sells hammers for $3.00, flashlights for $5, and Lanterns for $7.00. Store A sold 10 hammers, 2 flashlights and 2 lanterns. Store B sold 9 hammers, 14 flashlights and 5 lanterns. Store C sold 8 hammers, 6 flashlights and 7 lanterns.
    a. Create a matrix P for the Prices and a separate one, N, for the Number of Items sold per store.
    b. Find the product of the two matrices and explain in complete sentences what the product of the two matrices represents.
    c. Create a new matrix F that you can use along with P or N from part a to find the gross revenue from just the flashlights sold at all three stores.
    d. Suppose the owner plans to pay a 10% bonus for flashlight sales. How much total bonus can he expect to pay the employees?

2. The measure of the largest angle of a triangle is twice the measure of the smallest angle. The sum of the smallest angle and the largest angle is twice the other angle. Find the measure of each angle.

Warm Up Day 4 Answers

1. A chain of hardware stores sells hammers for $3.00, Flashlights for $5, and Lanterns for $7.00. Store A sold 10 hammers, 2 flashlights and 2 lanterns. Store B sold 9 hammers, 14 flashlights and 5 lanterns. Store C sold 8 hammers, 6 flashlights and 7 lanterns.

a. Create a matrix for the Prices and a separate one for the # of Items sold per store.

$$N = \begin{array}{c} \\ Hammers \\ Flashlights \\ Lanterns \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 10 & 9 & 8 \\ 2 & 14 & 6 \\ 2 & 5 & 7 \end{bmatrix} \end{array}$$

$$P = Price\,(\$) \begin{array}{c} \begin{array}{ccc} Hamm. & Flash. & Lantern \end{array} \\ \begin{bmatrix} 3.00 & 5.00 & 7.00 \end{bmatrix} \end{array}$$

b. Find the product of the two matrices and explain in complete sentences what the product of the two matrices represents.

	A	B	C
Revenue = PN = Price ($)	**[54**	**132**	**103]**

This product represents the total revenue for stores A, B, and C from hammers, flashlights, and lanterns.

c. Create a matrix F that you can use with P or N from part a to find the total gross revenue from the flashlights sold at all three stores.

$$F = Price\,(\$) \begin{array}{c} \begin{array}{ccc} H & Flash. & Ln \end{array} \\ \begin{bmatrix} 0 & 5.00 & 0 \end{bmatrix} \end{array}$$

	A	B	C
FN ($) =	**[10**	**70**	**30]**

Warm Up Answers

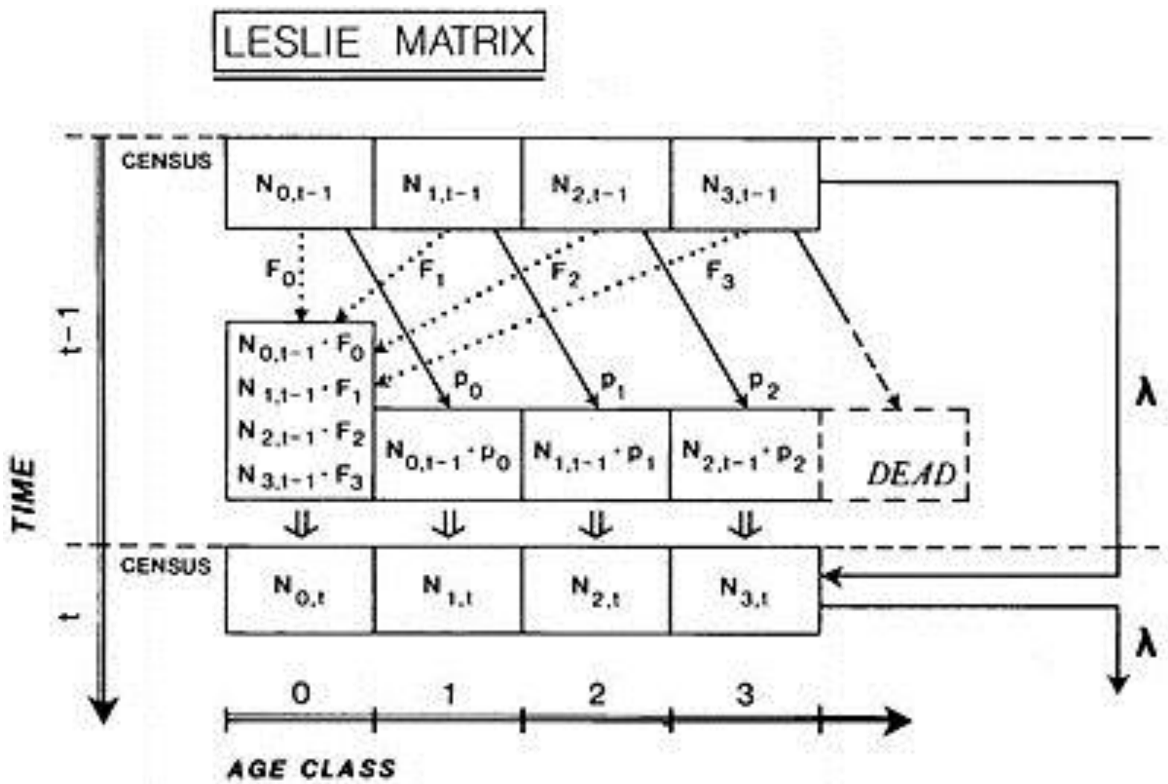
2. The measure of the largest angle of a triangle is twice the measure of the smallest angle. The sum of the smallest angle and the largest angle is twice the other angle. Find the measure of each angle.

40, 60, 80

Tonight's Homework

- Finish Packet p. 3 except #9
- Packet p. 4 #2a-d
  (about the deer = top half of p4)

Notes Day 4
Population Growth:
The Leslie Model (Part 1)
Section 3.4



Population Growth

- The management of wildlife, forests, fisheries as well as biological research are areas where it is important to understand rates of population growth.

- The Leslie Model is a commonly used tool for modeling age-specific population growth.
  - It was developed in 1945 by a British mathematician named P. H. Leslie, who worked at the Bureau of Animal Population in Oxford, England.
  - It is interesting to note that when he was first working with this model, there were no computers or calculators available. All computations were done by hand and some of the calculations took hours or days.

Population Growth

- For each age class, we need to know the reproduction rate and the rate of survival into the next age class.

- In order to simplify the model, we only consider the female populations (since only they reproduce). We can assume that the number of births and survival rates for male offspring would be the same as for female offspring.

Rattus Norvegicus

The following table lists reproduction and survival rates for the female population of a certain species of small brown rats. These rats reproduce every three months and have their 1st litter at approximately 3 months. Their life span is about 18 months, which gives a natural stopping point for the age classes.

Age (months)	Birthrate	Survival Rate	Initial Population of Rats
0-3	0	0.6	15
3-6	0.3	0.9	9
6-9	0.8	0.9	13
9-12	0.7	0.8	5
12-15	0.4	0.6	0
15-18	0	0	0

Copy down OR snap a pic of this table… we'll refer to it A LOT today!

Total Initial Population: 42 (find sum)

Age (months)	Birthrate	Survival Rate	Initial Population of Rats
0-3	0	0.6	15
3-6	0.3	0.9	9
6-9	0.8	0.9	13
9-12	0.7	0.8	5
12-15	0.4	0.6	0
15-18	0	0	0

How many females will there be after one 3-month cycle?

First, we have to find the number of new female babies (newborns).

How can we determine that from the information we have?

***Multiply the birthrate and the initial population for each age group and find the sum.**

$$15(0)+9(0.3)+13(0.8)+5(0.7)+0(0.4)+0(0)=16.6$$

The number of new baby rats to go into the 0 – 3 age group is around 16.6. But don't round to a whole number! It can make a big difference in calculations over time.

Continued…

How many females will there be after one 3-month cycle?

Second, we have to find how many female rats survive to move up to the next age group. **How can we determine that from this data?**

Age	Survival Rate	Initial Population	Number moving up to next age group
0-3	0.6	15	**9**
3-6	0.9	9	**8.1**
6-9	0.9	13	**11.7**
9-12	0.8	5	**4.0**
12-15	0.6	0	**0**
15-18	0	0	**0**

***Multiply the survival rate and the initial population for each age group to complete the distribution.**

What is the female population distribution after 3 months?

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	**16.6**	**9**	**8.1**	**11.7**	**4.0**	**0**

The newborns (from last slide)

The survivors are 3 months older

Total Population after one cycle:
49.4
(find sum)

Continued…

How many females will there be after one 3-month cycle?

Second, we have to find how many female rats survive to move up to the next age group. **How can we determine that from this data?**

Age (months)	Birth-rate	Survival Rate	Initial Population of Rats	**Number moving up to next age group**
0-3	0	0.6	15	**9**
3-6	0.3	0.9	9	**8.1**
6-9	0.8	0.9	13	**11.7**
9-12	0.7	0.8	5	**4.0**
12-15	0.4	0.6	0	**0**
15-18	0	0	0	**0**

***Multiply the survival rate and the initial population for each age group to complete the distribution.**

What is the female population distribution after 3 months?

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	**16.6**	**9**	**8.1**	**11.7**	**4.0**	**0**

The newborns (from last slide)

The survivors are 3 months older

Total Population after one cycle:
49.4
(find sum)

Summary

- 1st – Births – Find the number of newborns.
- 2nd – Survivors – How many move up to the next age group?
- 3rd – Build the new population array.
  - Newborns go into the first spot
  - Survivors move up to the next spot
- 4th – Add up the number in each age group to get the new total population.

Example 2

How many females will there be after 6-months?

Use Same Rat Population…

Age (months)	Birthrate	Survival Rate	Initial Population of Rats
0-3	0	0.6	15
3-6	0.3	0.9	9
6-9	0.8	0.9	13
9-12	0.7	0.8	5
12-15	0.4	0.6	0
15-18	0	0	0

Population distribution after 3 months (1 cycle). **(from last slide)**

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	16.6	9.0	8.1	11.7	4.0	0

1st) How many new babies after 6 months (2 cycles)?

$$16.6(0)+9(0.3)+8.1(0.8)+11.7(0.7)+4(0.4)+0(0)= \mathbf{18.97}$$

Multiply amount in 3 month age group by their birthrate.

Ex 2 Cont… How many females after 6-months?

Population distribution after 3 months (1 cycle).

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	16.6	9.0	8.1	11.7	4.0	0

b) How many survivors to move up to the next age group?

Age	Population	Survival Rate	Number moving up to next age group
0-3	16.6	0.6	**9.96**
3-6	9	0.9	**8.1**
6-9	8.1	0.9	**7.29**
9-12	11.7	0.8	**9.36**
12-15	4	0.6	**2.4**
15-18	0	0	**0**

***Multiply the survival rate and the initial population for each age group to complete the distribution.**

c) New distribution after 6 months (2 cycles).

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	**18.97** babies	9.96	8.1	7.29	9.36	2.4

Total Population After two cycles: 56.08

Example 2 Continued…

d) Distribution after 9 months (3 cycles).

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	18.32	11.38	8.96	7.29	5.83	5.62

Total Population
After three cycles:
57.4

Distribution after 12 months (4 cycles).

Age	0-3	3-6	6-9	9-12	12-15	15-18
Number	18.02	10.99	10.24	8.06	5.83	3.50

Total Population
After four cycles:
56.64

e) Compare the original number of rats with the numbers after 3, 6, 9, and 12 months. What do you observe?

42, 49.4, 56.08, 57.4, 56.64

f) What do you think might happen to this population if you extended the calculations to 15, 18, 21…months?

It will fluctuate then stabilize

Practice

Packet p. 4 #2a-d

- Deer Problem

Population Growth: The Leslie Model (Part 2) Section 3.5

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{juveniles} \\ \text{young adults} \\ \text{adults} \\ \text{old adults} \end{bmatrix}$$

Population Growth

- We have learned to use an initial population distribution, birth rates, and survival rates to predict population figures at future times.

- Looking a few cycles into the future is not impossible, but the arithmetic quickly becomes very cumbersome.

  What if a wildlife manager or urban planner wants to predict 20 or more cycles into the future?

  In the Packet problems, you will practice using matrices to hold the key to efficiently working with population data and predictions.

Population Growth

Let's return to our rat model and start with an Initial Population matrix P_0

$$P_0 = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix}$$

Recall that with a matrix, we could easily calculate the number of **newborns**.

$$\begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0.3 \\ 0.8 \\ 0.7 \\ 0.4 \\ 0 \end{bmatrix}$$

This yields a 1 number answer

1x6 X 6x1
= 1x1

Notice that we did not include the last column matrix that is full of 0s!

Recall also that with a matrix, we could easily calculate the number of **survivors**.

$$\begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0.6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.9 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.8 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.6 \\ 0 \end{bmatrix}$$

Here we want a list of answers

1x6 X 6x6
= 1x6

Leslie Model

We can combine our birth rates matrix and our survival rates matrices into a single matrix we will call the LESLIE MATRIX (L)

$$L = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The birth rates are in the first column.

The survival rates are in the Super Diagonal.

Leslie Model

Enter the Initial Population (P_0) and Leslie Matrix (L) into your calculator.

Multiply the Initial Population by the Leslie Matrix: P_0 L

The result is P_1 , the population distribution after one cycle.

$$P_0 L = P_1$$

$$\begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16.6 & 9 & 8.1 & 11.7 & 4 & 0 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 16.6 & 9 & 8.1 & 11.7 & 4 & 0 \end{bmatrix}$$

Leslie Model

$$P_1 = P_0\, L = \begin{bmatrix} 16.6 & 9 & 8.1 & 11.7 & 4 & 0 \end{bmatrix}$$

So, if we multiply P_1 by L, that will give us P_2, the population after 2 cycles.

$$P_2 = (P_1)(L)$$

Well, isn't that just $P_2 = (P_0\, L)(L)$ **?**

And, isn't that just the same as $P_2 = P_0\, L^2$ **?**

Try entering that into your calculator and see if you get the rat population distribution after 2 cycles.

What if you wanted to find the population distribution after 8 cycles?

$$P_8 = P_0\, L^8$$

$$P_8 = \begin{bmatrix} 21.03 & 12.28 & 10.9 & 9.46 & 7.01 & 4.27 \end{bmatrix}$$

The population distribution after 8 cycles.

Leslie Model

There's also an easy way with matrices to calculate the Total Population.

Enter a 6-element column matrix with all ONES into your calculator.

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Multiply P_8 by the column matrix (C).

$$(P_0 \bullet L^8) \bullet C$$

$$\begin{bmatrix} 21.03 & 12.28 & 10.9 & 9.46 & 7.01 & 4.27 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 64.95 \end{bmatrix}$$

The total population after 8 cycles is 64.95 rats

Ended here Fall ‘18

- Next slides left for following day…

Exercise #2

Suppose the rats start dying off from overcrowding when the total female population for a colony reaches 250. Find how long it will take for this to happen when the initial population distribution is:

$a)\begin{bmatrix} 18 & 9 & 7 & 0 & 0 & 0 \end{bmatrix}$

a) 60 cycles

$b)\begin{bmatrix} 35 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

You Try!

b) 69 cycles

Age (months)	Birthrate	Survival Rate	Initial Population of Rats
0-3	0	0.6	15
3-6	0.3	0.9	9
6-9	0.8	0.9	13
9-12	0.7	0.8	5
12-15	0.4	0.6	0
15-18	0	0	0

Growth Rates Between Cycles

STILL working with the rats.
Consider the table below.

$$P_0 = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix}$$

Cycle	Total Population	Growth Rate
Original	42	
1	49.4	17.6%
2	56.08	13.5%
3	57.40	2.4%
4	**56.65**	**–1.31%**
5	**59.35**	**4.77%**
6	**61.76**	**4.06%**

The general formula for finding the growth rate between cycles is:

$$G.R. = \frac{(new - previous)}{previous}$$

To find the growth rate between the initial population and the first cycle we would do the following calculation.

$$\frac{(49.4 - 42)}{42} \approx .176$$

Notice the growth rates appear to decline then increase again.

Growth Rates Between Cycles

Find the growth rates between P_{19} and P_{20}, between P_{20} and P_{21}, between P_{25} and P_{26}, between P_{26} and P_{27}

$$P_0 = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix}$$

Cycle	Total Population	Growth Rate
19	90.627	
20	93.384	**3.04%**
21	**96.230**	**3.05%**
25	**108.488**	
26	**111.789**	**3.04%**
27	**115.191**	**3.04%**

One characteristic of the Leslie Model is that the growth does stabilize at a rate called the *LONG-TERM GROWTH RATE* of the population.

The long-term growth rate for this rat population is 3.04% or .0304.

Exercise #3

a) Find the long-term growth rate of the total population for each of the initial population distributions in #2 (From earlier today.)

$$a)\begin{bmatrix}18 & 9 & 7 & 0 & 0 & 0\end{bmatrix}$$

a) 3.04%

$$b)\begin{bmatrix}35 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

b) 3.04%

b) How does the initial population distribution seem to affect the long-term growth rate?

The initial population distribution does NOT seem to affect the long term growth rate.

Leslie Model – A Special Case

A special case that P.H. Leslie studied involved creatures that reproduce only in one phase of their life cycle.

Consider the scenario posed below. This certain kind of bug lives only 3 weeks. 50% of the bugs survive from the first week of life to the second. 70% of those who make it to the second week also survive into the third. No bugs live beyond three weeks. On average 6 newborn bugs are produced by those bugs who make it to the third week. Create a table from info.

Use this description to construct a Leslie Matrix for the life cycle of this bug.

Weeks	Birth Rate	Survival Rate
1	0	.5
2	0	.7
3	6	0

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Special Case Practice

A group of 5 3-week old female bugs move into your basement.

What is P_0 ?

$$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

How long will it be before the bug population passes 1,000 bugs ?

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Answer on the next slide…

Leslie Model – A Special Case

How long before there are at least 1,000 female bugs in the basement?

$$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Note: To use the iteration function on the graphing calculator:

```
[A]*[B]
[[30.0 0.0 0.0]]
Ans*[B]
[[0.0 15.0 0.0]]
[[0.0 0.0 10.5]]
[[63.0 0.0 0.0]]
[[0.0 31.5 0.0]]
```

Press [A] [x] [B] and press [ENTER].

Press [x] [B] and press [ENTER].

Continue to press [ENTER] while you count the cycles.

Use this iteration function to observe the pattern. Could you put this information into a table ?

There will be 1,225 female bugs living in the basement after 16 weeks.

Leslie Model – A Special Case

Make a table of the population distributions for cycles P_{22} through P_{30}.

$$P_0 = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.7 \\ 6 & 0 & 0 \end{bmatrix}$$

Cycle	One Week	Two Weeks	Three Weeks
0	0	0	5
22	**5,403.3**	**0**	**0**
23	**0**	**2,701.6**	**0**
24	**0**	**0**	**1,891.1**
25	**11,346.9**	**0**	**0**
26	**0**	**5,673.4**	**0**
27	**0**	**0**	**3,971.4**
28	**23,828.4**	**0**	**0**
29	**0**	**11,914.2**	**0**
30	**0**	**0**	**8,339.9**

a) Do you observe a pattern?

What about the growth rate? How should we calculate it?

b) Examine the population change from :

P_{22} to P_{25} ,
P_{23} to P_{26} ,
P_{24} to P_{27} ,
P_{25} to P_{28} ,
P_{26} to P_{29} ,
P_{27} to P_{30}
Make a conjecture.

Tonight's Homework

- Finish Packet p. 3 except #9
- Packet p. 4 #2a-d
  (about the deer = top half of p4)

Extra problem on next slide…

Warm Up Answers

3. Solve the system using matrices.

$$\begin{aligned} x+2y &= 2z-6 \\ y+z &= 2 \\ -2x-y+3z &= 4 \end{aligned} \qquad \begin{aligned} x+2y-2z &= -6 \\ y+z &= 2 \\ -2x-y+3z &= 4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ -2 & -1 & 3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$$

$$x = 4, \quad y = -\frac{3}{2}, \quad z = \frac{7}{2}$$